GESTALTEN

Day 1

Overview talk (Aoki, 60 min).

Lecture 1: Basics of presentable $(\infty,1)$-categories (60 min). Reference: [Sch26, Lecture II]

In general, ∞-categories that we would like to consider (e.g., the derived $(\infty,1)$-category $D(\mathbb{Z})$ of $\mathbb{Z}$-modules) are not essentially small, and there is a size issue. However, the $(\infty,1)$-category $D(\mathbb{Z})$ is controlled by the small subcategory $\mathrm{Perf}(\mathbb{Z})$ of perfect complexes. Such ∞-categories are called *presentable* $(\infty,1)$*-categories*, and they play an important role in the theory of gestalten. More precisely, for each regular cardinal κ, we define the notion of κ*-presentable* $(\infty,1)$*-categories*. The key property of κ-presentable $(\infty,1)$-categories is that the $(\infty,1)$-category Pr^L_κ of κ-presentable $(\infty,1)$-categories is κ-presentable. In other words, we have $\mathrm{Pr}^L_\kappa \in \mathrm{Pr}^L_\kappa$.

Outline:

(1) Define κ-presentable $(\infty,1)$-categories and related notions (e.g., κ-compact objects, $\mathrm{Ind}_\kappa(-)$). See also [Lur09, 5.3, 5.4] or [Kerodon, 04KE].
(2) Explain the adjoint functor theorem ([Sch26, Theorem 3.1 and Corollary 3.2]).
(3) Sketch of the proof of $\mathrm{Pr}^L_\kappa \in \mathrm{Pr}^L_\kappa$ ([Sch26, Theorem 2.9]). See also [Aok25, 2.1].

Lecture 2: Algebras in Pr^L (90 min). Reference: [Sch26, Lecture III].

In [Lur09, Lur17], Lurie gives a very convenient framework to do "algebra" in symmetric monoidal $(\infty,1)$-categories, such as $D(\mathbb{Z})$. On the other hand, Pr^L_κ admits a natural symmetric monoidal structure (Lurie's tensor product). Therefore, we can study symmetric monoidal κ-presentable $(\infty,1)$-categories using "algebra".

As explained in Lecture 1, we have $\mathrm{Pr}^L_\kappa \in \mathrm{Pr}^L_\kappa$, and we can regard Pr^L_κ as a commutative algebra object in Pr^L_κ. Thus, we can define the $(\infty,1)$-category $2\mathrm{Pr}^L_\kappa \coloneqq \mathrm{Mod}_{\mathrm{Pr}^L_\kappa}(\mathrm{Pr}^L_\kappa)$. For $C \in 2\mathrm{Pr}^L_\kappa$, we have the Pr^L_κ-module structure $\mathrm{Pr}^L_\kappa \otimes C \to C$, and as a right adjoint, we get $\mathrm{Hom}_C(x,y) \in \mathrm{Pr}^L_\kappa$ for $x,y \in C$. In other words, we can regard C as a category enriched in (κ-presentable) $(\infty,1)$-categories (i.e., κ-presentable "$(\infty,2)$-category") and $2\mathrm{Pr}^L_\kappa$ as the $(\infty,1)$-category of "(κ-presentable) $(\infty,2)$-categories". Inductively, we define the $(\infty,1)$-category $n\mathrm{Pr}^L_\kappa$ of "(κ-presentable) (∞,n)-categories" by $n\mathrm{Pr}^L_\kappa \coloneqq \mathrm{Mod}_{(n-1)\mathrm{Pr}^L_\kappa}(\mathrm{Pr}^L_\kappa)$. From now on, we take $\kappa = \aleph_1$[1] and abbreviate $n\mathrm{Pr} = n\mathrm{Pr}^L_{\aleph_1}$. Roughly speaking, $C_n \in \mathrm{CAlg}(n\mathrm{Pr})$

[1] In practice, this choice does not affect concrete mathematical arguments.

is an (∞, n)-category. Moreover, each object $D \in C_n$ defines a ($\aleph_1$-presentable) $(\infty, n-1)$-category $\mathrm{Hom}_{(n-1)\mathrm{Pr}}(\mathbf{1}, D) \in (n-1)\mathrm{Pr}$, and using this, we can regard $D \in C_n$ as an $(\infty, n-1)$-category. We define a Stefanich ring as a sequence $C_1 \in C_2 \in \cdots$ of symmetric monoidal $\aleph_1$-presentable (∞, n)-categories such that $C_n \in C_{n+1}$ is a symmetric monoidal unit of C_{n+1}, and we define the $(\infty, 1)$-category StRing of Stefanich rings by $\mathrm{StRing} = \lim \mathrm{CAlg}(n\mathrm{Pr})$.

Outline:

(1) Explain limits and colimits in Pr^L and Pr^R ([Sch26, 3.1, 3.2]).
(2) Explain Lurie's tensor product and show examples ([Sch26, Definition/Proposition 3.6, Example 3.7, Proposition 3.8, and Example 3.12])
(3) For a symmetric monoidal κ-presentable $(\infty, 1)$-category C, explain a fully faithful functor

$$\mathrm{CAlg}(C) \to \mathrm{CAlg}(\mathrm{Mod}_C(\mathrm{Pr}^L_\kappa));\ A \mapsto \mathrm{Mod}_A(C),$$

and its right adjoint functor $D \mapsto \mathrm{End}_D(\mathbf{1})$ ([Sch26, Theorem 3.11]).
(4) Define the $(\infty, 1)$-category StRing of Stefanich rings ([Sch26, Definition 3.13]).

Lecture 3: Dualizable categories (60 min). Reference: [Sch26, Appendix to Lecture IV].

As in usual theory of algebra, the notion of dualizable objects in $\mathrm{Mod}_{\mathcal{V}}(1\mathrm{Pr})$ is important in higher category theory (cf. [Efi25]). Ramzi's paper [Ram24] is a good reference for dualizable categories.

Outline:

Review the results in Appendix to Lecture IV in [Sch26], especially [Sch26, Theorem 4.13, Definition 4.20, 4.21, Theorem 4.22].

Day 2

Lecture 4: Quasi-coherent sheaves of higher categories, I (60 min). Reference: [Sch26, Lecture IV].

In this lecture and the next lecture, as a warm up for Lecture 6, we study Stefanich rings obtained from $\mathrm{CAlg}(D(\mathbb{S}))$. For $A \in \mathrm{CAlg}(D(\mathbb{S}))$, let $D(A)$ denote the category of A-modules. For $n \geq 1$, we define inductively $n\mathrm{Pr}_A = \mathrm{Mod}_{(n-1)\mathrm{Pr}_A}(1\mathrm{Pr})$, where $0\mathrm{Pr}_A = D(A)$. Then $(D(A), 1Pr_A, 2\mathrm{Pr}_A, \ldots)$ is a Stefanich ring, and we get a functor $\mathrm{CAlg}(D(\mathbb{S})) \to \mathrm{StRing}$[2]. For $f\colon A \to B$ in $\mathrm{CAlg}(D(\mathbb{S}))$, we write $f_n^*\colon n\mathrm{Pr}_A \to n\mathrm{Pr}_B$. The functor $f_0^*\colon D(A) \to D(B)$ has a $D(A)$-linear[3] right adjoint functor $f_{0*}\colon D(B) \to D(A)$, and if B is a countably presented A-algebra, then the functor $f_{0*}\colon D(B) \to D(A)$ defines a morphism in $1\mathrm{Pr}_A$[4]. However, in general, the functor $f_0^*\colon D(A) \to D(B)$ does not necessarily have a left adjoint functor. A surprising

[2]Note that $n\mathrm{Pr}_A$ is an object of $\mathrm{CAlg}((n+1)\mathrm{Pr})$. Therefore, we can regard $n\mathrm{Pr}_A$ as an $(\infty, n+1)$-category, and an object $C \in n\mathrm{Pr}_A$ as an (∞, n)-category.

[3]$D(A)$-linearity of the functor f_{0*} means that f_{0*} satisfies the projection formula.

[4]If B is a countably presented A-algebra, then the functor f_{0*} preserves $\aleph_1$-compact objects.

point in higher category theory is that the situation is much simplified in the case of $n \geq 1$. More precisely, we have the following theorem:

Theorem 0.1 ([Sch26, Corollary 4.6])**.** *Let $f\colon A \to B$ be a countably presented map in $\mathrm{CAlg}(D(\mathbb{S}))$.*

(1) *For $n \geq 1$, the functor $f_n^*\colon n\mathrm{Pr}_A \to n\mathrm{Pr}_B$ has a left adjoint $f_{n\sharp}\colon n\mathrm{Pr}_B \to n\mathrm{Pr}_A$ and a right adjoint $f_{n*}\colon n\mathrm{Pr}_B \to n\mathrm{Pr}_A$ in $(n+1)\mathrm{Pr}_A$*[5]*.*
(2) *There is a canonical identification $f_{n*} \cong f_{n\sharp}$.*
(3) *For $n \geq 0$, $n\mathrm{Pr}_B \in (n+1)\mathrm{Pr}_A$ is dualizable.*

This phenomenon, of functors having identical left and right adjoints, is called *ambidexterity*. Ambidexterity is a unique phenomenon in higher category theory, and it will play an important role in the following lectures.

Finally, by using Mathew's descendability argument, we find that the functor $\mathrm{CAlg}(D(\mathbb{S})) \to \mathrm{StRing}$ is a hypersheaf in the fpqc topology. Therefore, we can extend the above construction to fpqc stacks, and we get a functor[6]

$$\mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{CAlg}(D(\mathbb{S}))) \to \mathrm{StRing};\ X \mapsto (D(X), 1\mathrm{Pr}_X, 2\mathrm{Pr}_X, \ldots).$$

Outline:

(1) Define a functor $\mathrm{CAlg}(D(\mathbb{S})) \to \mathrm{StRing}$, and construct a right adjoint functor $f_{n*}\colon n\mathrm{Pr}_B \to n\mathrm{Pr}_A$ of $f_n^*\colon n\mathrm{Pr}_A \to n\mathrm{Pr}_B$ for $f\colon A \to B$ in $\mathrm{CAlg}(D(\mathbb{S}))$ ([Sch26, Proposition 4.1]).
(2) Explain "ambidexterity" and prove Theorem 0.1 ([Sch26, 4.2]).
(3) Review the notion of Mathew's descendability ([Mat16]), and sketch the proof of fpqc descent ([Sch26, Theorem 4.8]).

Lecture 5: Quasi-coherent sheaves of higher categories, II (90 min). Reference: [Sch26, Lecture V].

In this lecture, we extend Theorem 0.1 from affine schemes to stacks. The main theorem in this lecture is the following:

Theorem 0.2 ([Sch26, Theorem 5.2])**.** *Let $f\colon Y \to X$ be a morphism (satisfying some countability conditions) in $\mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{CAlg}(D(\mathbb{S})))$.*

(1) *For $n \geq 1$, the functor $f_n^*\colon n\mathrm{Pr}_X \to n\mathrm{Pr}_Y$ has a left adjoint $f_{n\sharp}$ in $(n+1)\mathrm{Pr}_X$.*
(2) *For $n \geq 2$, then the functor $f_{n\sharp}$ is also a right adjoint of f_n^* in $(n+1)\mathrm{Pr}_X$.*
(3) *For $n \geq 1$, $n\mathrm{Pr}_Y \in (n+1)\mathrm{Pr}_X$ is dualizable.*

When $n = 0$, the functor $f_0^*\colon D(X) \to D(Y)$ usually does not admit a left adjoint $f_{0\sharp}$ in $1\mathrm{Pr}_X$. In terms of 6-functor formalisms, if f is "étale", then f_0^* admits a left

[5] We regard the (∞, n)-category $n\mathrm{Pr}_B$ as $f_{n+1,*}\mathbf{1} \in (n+1)\mathrm{Pr}_A$, where $f_{n+1,*}\colon (n+1)\mathrm{Pr}_B \to (n+1)\mathrm{Pr}_A$ is a right adjoint of the functor $f_{n+1}^*\colon (n+1)\mathrm{Pr}_A \to (n+1)\mathrm{Pr}_B$ of presentable $(\infty,1)$-categories.

[6] In general, this $D(X)$ might be different from the usual $D(X)$ because of the difference of limits in $1\mathrm{Pr} = \mathrm{Pr}^L_{\aleph_1}$ and Pr^L. However, if X is a countable colimits of affine schemes, then there is no difference between these two $D(X)$.

adjoint in $1\mathrm{Pr}_X$. Similarly, when $n=0$, the functor $f_0^*\colon D(X)\to D(Y)$ usually does not admit a aright adjoint f_* in $1\mathrm{Pr}_X$, and if f is "proper", then f_0^* admits a right adjoint in $1\mathrm{Pr}_X$. Therefore, roughly speaking, Theorem 0.2 claims that for $n\geq 1$ "all maps are étale", while for $n\geq 2$ "all maps are proper".

Outline:

Explain the details of [Sch26, Theorem 5.2, Corollary 5.3, 5.4, 5.5].

Lecture 6: Affine, Proper/Prim, and Étale/Suave (150 min). Reference: [Sch26, Lecture VI].

In this lecture, we study the phenomena appearing in the previous two lectures more systematically for arbitrary Stefanich rings.

Outline:

(1) Explain the construction in [Sch26, Proposition 6.2].
(2) Define the notion of n-*affine*, and prove their basic properties ([Sch26, Proposition 6.6]).
(3) Define the notion of n-*prim* and n-*proper*, and prove their basic properties ([Sch26, Proposition 6.10, Proposition 6.13, Lemma 6.15]). Explain the relationship between 0-properness and rigidity in the sense of Gaitsgory-Rozenblyum ([Sch26, Proposition 6.14]). Assuming [Sch26, Corollary 8.14], sketch the proof of [Sch26, Proposition 8.1].
(4) Define the notion of n-*suave* and n-*étale*, and prove their basic properties ([Sch26, Proposition 6.18, Proposition 6.20]).
(5) Explain the relationship between *primness* and *suaveness* (ambidexterity, [Sch26, Proposition 6.21, Lemma 6.22]).

Day 3

Lecture 7: Gestalten (90 min). Reference: [Sch26, Lecture VII].

We define the $(\infty,1)$-category Gest of gestalten by $\mathrm{Gest}:=\mathrm{StRing}^{\mathrm{op}}$, and we write an object in Gest corresponding to $A\in\mathrm{StRing}$ as $\mathrm{Gest}(A)$. The main theorem in this lecture is the following:

Theorem 0.3 ([Sch26, Theorem 7.1])**.** *Let A be a Stefanich ring, and $n\geq 0$. Let $\mathrm{StRing}_{A/}^{n\text{-et}}$ be the $(\infty,1)$-category of n-étale A-algebras. Then $(\mathrm{StRing}_{A/}^{n\text{-et}})^{\mathrm{op}}$ is a countably $(\infty,1)$-topos. In other words, $\mathrm{Ind}_{\aleph_1}(\mathrm{StRing}_{A/}^{n\text{-et}})^{\mathrm{op}}$ is an $(\infty,1)$-topos*[7]*.*

Outline:

(1) Review (countably) $(\infty,1)$-topoi, especially Giraud's axiom ([Sch26, Theorem 7.2, Proposition 7.3]).
(2) Prove [Sch26, Proposition 7.5].
(3) Prove Theorem 0.3 ([Sch26, Theorem 7.1, Theorem 7.7]).
(4) Explain the descent of affineness (resp. primness, properness, suaveness, étaleness) ([Sch26, Proposition 7.9]).

[7] In practice, we can work with a sufficiently large fixed n (e.g., $n=10$).

Lecture 8: Descendability in higher categories (90 min). Reference: [Sch26, Appendix to Lecture VIII].

In this lecture, we give a criterion for whether a map $f\colon X \to Y$ in Gest is surjective in practical situations. This criterion can be regarded as a generalization of Mathew's descendability to higher categories.

Outline:

(1) Prove [Sch26, Proposition 8.6, 8.7, 8.8, 8.10, 8.11, Lemma 8.13, Corollary 8.14].
(2) Explain the relation with the notion of Mathew's descendability ([Sch26, Corollary 8.15, 8.16]).

Lecture 9: Examples (90 min). Reference: [Sch26, Lecture VIII, IX].

In this lecture, we give examples of Gestalten (e.g., $\mathbb{A}^1$, $\mathbb{G}_a$, GL_1, $\mathbb{G}_m$, $\mathbb{P}^1$, $*/\mathbb{G}_m$, X_{dR}). It is also possible to construct gestalten from a general 6-functor formalism.

Outline:

(1) Briefly introduce examples in [Sch26, Lecture VIII]. Note that [Sch26, Proposition 8.1] was already explained in Lecture 6.
(2) Explain the construction of gestalten from a general (small) 6-functor formalism ([Sch26, Propostion 9.3, 9.4, 9.5]). Briefly explain the extension of the above construction to the stacks ([Sch26, Proposition 9.7, Corollary 9.8]).

Day 4

Lecture 10: Cartier duality (120-180 min). Reference: [Sch26, Lecture X, XI, XII].

Advanced topics, I (Aoki, 90-120 min).

Day 5

Advanced topics, II (Aoki, 150-180 min).

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