博士論文

A six-functor formalism for sheaves over locally proper maps and the Bauer–Furuta construction

（局所固有写像上の層に対する六つの関手の枠組み，そして Bauer–Furuta の構成）

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A SIX-FUNCTOR FORMALISM FOR SHEAVES OVER LOCALLY PROPER MAPS AND THE BAUER–FURUTA CONSTRUCTION

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Abstract. We present a new construction of the Bauer–Furuta invariant as well as its family version, which avoids the conventional use of finite-dimensional approximation arguments and instead relies on sheaf-theoretic methods. To this end, we develop a six-functor formalism for sheaves of spectra on topological spaces, in which separated locally proper maps admit exceptional shriek functors. This framework enables us to define Borel–Moore-type homology spectra relative to C^1-differentiable Fredholm maps between Banach manifolds, whose dualizing objects are given by the Thom spectra of the Atiyah–Singer families index. As a result, our construction of the Bauer–Furuta invariant naturally extends to families over more general base spaces.

Contents

1. Introduction

The *Bauer–Furuta invariant* [BF04], a stable homotopy-theoretic refinement of the Seiberg–Witten invariant defined for closed spin^c 4-manifolds, is constructed by reducing an infinite-dimensional gauge-theoretic setting to a finite-dimensional approximation and employing explicit models of stable homotopy theory of spectra. In this paper, we present a new construction of the invariant that avoids such reductions entirely. Our approach expresses the Bauer–Furuta invariant as a morphism (between sheaves of spectra) of the form

$$r_*f^!(\mathbf{1}) \longrightarrow q_*(\mathbf{1}) \tag{1.1}$$

within a six-functor formalism for sheaves of spectra on topological spaces, where f is a proper map, $r = qf$ is the composite with the projection onto a base space, and $\mathbf{1}$ is the constant sheaf at the sphere spectrum. The situation is depicted as follows.

$$\begin{tikzcd} Y' \arrow[rr, "f"] \arrow[dr, "r"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd} \tag{1.2}$$

This reformulation not only simplifies the construction, but also places the invariant within a broader context that provides its natural generalization to families and a higher categorical functoriality.

The key novelty lies in extending the six-functor formalism for sheaves, which was originally developed only for locally compact Hausdorff spaces, to a setting where the domain and codomain of a proper map f may be infinite-dimensional and locally non-compact. In particular, our formalism applies to the spaces that fit into the domain or codomain of a nonlinear elliptic PDE on a compact manifold, which is of main interest in gauge theory.

Moreover, unlike traditional homotopy theory, it is crucial to work beyond locally constant sheaves of spectra and deal with all sheaves of spectra, though the resulting Bauer–Furuta invariant does take values in locally constant objects. This is because the spaces on which the map f we will care is defined are often contractible, and thus the homotopy invariance of the ∞-category of locally constant sheaves implies that there will never be a nontrivial information that can be derived from them. Alternatively, it is also well-known that although the exceptional pullback functor $f^!$ often takes values in locally constant sheaves, there is no categorical argument to ensure the existence of the functor $f^!$ in the ∞-category of locally constant sheaves.

1.1. **The Bauer–Furuta theory.** To recall the theory of Bauer–Furuta invariant, let $(X, g, \mathfrak{s})$ be a closed riemannian spin^c 4-manifold equipped with a connection on the determinant line bundle. In the proof [Fur01] of the weaker version of *the* $\frac{11}{8}$*-conjecture*, Furuta used the Seiberg–Witten equation to obtain a $U(1)$-equivariant[1] proper map between $\mathbb{R}$-Hilbert spaces[2]

$$f\colon L^2_k(X; T^* \oplus S^+) \times H^0(X;\mathbb{R}) \longrightarrow L^2_{k-1}(X; S^- \oplus \Lambda^{2,+}T^* \oplus \Lambda^0 T^*) \tag{1.3}$$

and construct a $U(1)$-equivariant map

$$S^{2\mathrm{ind}_{\mathbb{C}}\not{D}+b_1(X)} \to S^{b_2^+(X)}$$

between (stable) spheres, where $\mathrm{ind}_{\mathbb{C}}$ denotes the complex Atiyah–Singer index. It can be better described as a map

$$\mathrm{BF}_f\colon \mathbb{S}^{2\mathrm{ind}_{\mathbb{C}}\not{D}} \to \mathbb{S}^{b_2^+(X)}$$

between (genuine) equivariant sphere spectra over the base space $\frac{H^1(X;\mathbb{R})}{H^1(X;\mathbb{Z})}$ as defined in [BF04]. The resulting map between sphere spectra was shown to be independent of the choices of metrics and connections. This is referred to as the *Bauer–Furuta invariant* of $(X, \mathfrak{s})$.

Our new definition of the Bauer–Furuta map is explained as follows. Since the Seiberg–Witten map (1.3), say $f\colon \mathcal{H}' \to \mathcal{H}$, is a proper map, there exists a functor $f^!$ right adjoint to the pushforward functor f_* defined between the stable ∞-categories of sheaves of spectra $\mathsf{Sh}(\mathcal{H}')$ and $\mathsf{Sh}(\mathcal{H})$. In particular, we have the counit map $f_*f^! \to \mathrm{id}$ in $\mathsf{Sh}(\mathcal{H})$, and by applying the global section functor to this counit map we obtain the proper pushforward or the transfer map (1.1), namely,

$$\mathrm{tr}_f\colon \Gamma(\mathcal{H}'; \omega_f) \to \Gamma(\mathcal{H}; \mathbb{S}).$$

Here, $\omega_f = f^!(\mathbf{1})$ is called the dualizing sheaf. We note that the domain of the transfer map can be regarded as a Borel–Moore-type homology relative to the map f.

1.2. **Main results.** Our first main theorem identifies this transfer map with the original Bauer–Furuta map. To state this, we briefly recall the construction of the Bauer–Furuta map. The argument presented below is referred to as the *finite-dimensional approximations*. Consider a (sufficiently large) finite-dimensional subspace $W \subset \mathcal{H}$. The linear part $l := df$ of the Seiberg–Witten map f has the property of being Fredholm, and defines a finite-dimensional subspace $W' = l^{-1}(W)$ of $\mathcal{H}'$. Using the orthogonal projection $\mathrm{pr}_W\colon \mathcal{H} \to W$, one can show that[3] there exists a bounded open neighborhood $N \subset \mathcal{H}'$ of the zero set $f^{-1}(0)$ such that the composite

$$f_W^{\mathrm{apprx}}\colon W' \cap \overline{N} \hookrightarrow \mathcal{H}' \xrightarrow{f} \mathcal{H} \xrightarrow{\mathrm{pr}_W} W \tag{1.4}$$

maps $W' \cap (\overline{N} - N)$ to $W - \{0\}$.[4] Thus, via the Pontrjagin–Thom collapse, we obtain a map on spheres

$$S^{W'} \xrightarrow{\text{collapse}} \frac{W' \cap \overline{N}}{W' \cap (\overline{N} - N)} \xrightarrow{f_W^{\mathrm{apprx}}} \frac{W}{W - \{0\}} \simeq S^W. \tag{1.5}$$

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[1] If $\mathfrak{s}$ comes from a spin structure, then it lifts to a Pin(2)-equivariant map.

[2] L^2_k is the Sobolev completions, $k > 4$, T^* is the cotangent bundle of X, $S^\pm$ is the positive/negative spinor bundles, and $\Lambda^{2,+}$ is the $(+1)$-eigenspace for the Hodge star operator on $\Lambda^2 T^*$.

[3] See [Fur01, Lemma 3.4] or [BF04, Lemma 2.3].

[4] f_W^{apprx} itself may not be a proper map.

Taking $\operatorname*{colim}_{W\to\infty}\Sigma^{\infty-W}$[5] on both sides, we get the desired map

$$\mathrm{BF}_f\colon \mathbb{S}^{\mathrm{ind}(df)} \to \mathbb{S}. \tag{1.6}$$

One of our main results is the following.

Theorem 1.2.1. *Let $f\colon \mathcal{H}' \to \mathcal{H}$ be the Seiberg–Witten map (1.3). As f is proper, it defines the transfer map $\mathrm{tr}_f\colon \Gamma(\mathcal{H}';\omega_f) \to \Gamma(\mathcal{H};\mathbb{S})$ between spectra-valued sheaf cohomology, where ω_f is the appropriate dualizing sheaf, and this transfer map can be identified with the Bauer–Furuta map (1.6). In particular, the both sides of the transfer map is equivalent to $\mathbb{S}^{\mathrm{ind}(df)}$ and $\mathbb{S}$, respectively.*

The original construction of the map (1.6) involves certain inconveniences, as some desirable properties, such as functoriality or independence of the choices of metrics, connections, or perturbations, are not so obvious. We note that this construction relies on a slightly stronger condition than the mere properness of the map (1.3); some quantitative control from functional analysis is implicitly used, which typically involves a specific concrete description of the map itself. The theorem above thus provides a solid argument for defining the Bauer–Furuta invariant in a completely canonical way. In fact, our construction of the transfer map relies only on the six-functor formalism and does not depend on any "bornological" property or functional analysis input or coordinate-level models, and thereby clarifies the functorial and categorical nature of the invariant from the outset. In particular, it proves that the Bauer–Furuta invariant depends only on the topological nature of the Seiberg–Witten map (1.3), which was not well-known before.

The Bauer–Furuta invariant, as previously mentioned, has been utilized in the partial progress toward the $\frac{11}{8}$-conjecture, a geography problem for closed 4-manifolds, which is most extensively studied in [HLSX22]. More recently, a family version of the Bauer–Furuta invariant has been employed to prove the existence of exotic diffeomorphisms as in [BK22] and subsequent works. However, the Bauer–Furuta invariants for families have yet to be studied in full generality. It is also worth noting that the Bauer–Furuta invariant should be part of a broader framework that resembles a *topological quantum field theory*. To ensure the $(\infty,0)$-functoriality of such a topological field theory, at least a comprehensive understanding of the family version of the Bauer–Furuta invariant is required. On the other hand, whether the original Bauer–Furuta construction generalizes to families over an arbitrary base space was nontrivial especially for noncompact non-CW base spaces.

Using the six-functor formalism, we also provide the family version of the Bauer–Furuta construction. This follows exactly the same definition as the map (1.1)—for a proper map f with any map q from the codomain of f with composite $r = qf$, the adjunction counit for $f^!$ defines the map $r_*f^! \to q_*$, which specializes to the following situation.

Theorem 1.2.2. *Let $\mathcal{H}'$ and $\mathcal{H}$ be Banach vector bundles over a topological space S. Given a proper map $f\colon \mathcal{H}' \to \mathcal{H}$ that is fiberwise C^1-differentiable Fredholm (whose differentials depends continuously on the base with respect to the norm topology), there exists the transfer map relative to the base S of the form*

$$\mathrm{tr}_f\colon \Sigma^{\mathrm{ind}(df)}\mathbf{1}_S \to \mathbf{1}_S$$

in the category $\mathsf{Sh}(S)$ of sheaves of spectra on S. Here, $\Sigma^{\mathrm{ind}(df)}$ is (tensoring with) the Thom spectrum sheaf of the families Atiyah–Singer index of the family of Fredholm operators df, and $\mathbf{1}_S$ is the constant sphere spectrum sheaf on S.

Since we aim to handle those infinite-dimensional manifolds such as Banach spaces, it is crucial to extend the six-functor formalism for sheaves which was originally developed for locally compact Hausdorff spaces to include locally proper maps between possibly locally non-compact spaces. To this end, we construct a six-functor formalism adapted to sheaves of spectra on topological spaces that are not necessarily locally compact, where the exceptional functors still exist for *(separated) locally proper* maps. Although the existence of such a six-functor formalism

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[5] Although the functoriality in W is not very clear at this point, these colimits actually stabilize at some W.

is known to experts in the field, we provide a streamlined and self-contained construction suited to our needs.

Theorem 1.2.3. *Let E denote the collection of separated locally proper maps (Theorem 3.1.1) between topological spaces.*

(1) *There exists a six-functor formalism, i.e., a lax symmetric monoidal functor*

$$\mathsf{Sh}(-)\colon \mathsf{Corr}(\mathsf{Top}; E) \to \mathsf{Pr}^{\mathrm{L}}_{\mathrm{st}}$$

where the left-hand side is the category of correspondences between topological spaces whose forward direction arrows are contained in E. It sends a topological space S to the category $\mathsf{Sh}(S)$ of sheaves of spectra on S and a backward direction arrow g to the usual pullback functor g^*.

(2) *Every C^1-class Fredholm map $f\colon L \to Y$ between Banach manifolds is an element of E. The dualizing sheaf $\omega_f \in \mathsf{Sh}(L)$ is given by the Thom spectrum (Theorem 2.1.8) of the linear differential df of f.*

It is also noticeable that most attempts to construct homotopy theoretic invariants for families have opted to use the *local systems* of spectra rather than sheaves of spectra. For our purpose, we find that the sheaf-based approach is significantly more natural and effective than local systems. This is particularly evident in cases involving locally proper maps between Banach spaces, where the lack of geometrically convenient compactifications makes understanding Borel–Moore homology through local systems highly impractical.

Remark 1.2.4. Due to [CLL25], we also obtain the $(\infty, 2)$-functor of the form

$$\mathsf{2Corr}(\mathsf{Top}; E, \mathrm{all})_{\mathrm{open},\mathrm{proper}} \to \mathsf{Pr}^{\mathrm{R}}_{\mathrm{st}}$$

sending a 2-correspondence consisting of separated locally proper maps f_0, f_1, an open embedding j, and a proper map p fitting into the following diagram:

$$\begin{tikzcd}
 & S_0 & \\
Y_0 \arrow[ur, "f_0"] \arrow[dr, "g_0"'] & W \arrow[l, "j"'] \arrow[r, "p"] & Y_1 \arrow[ul, "f_1"'] \arrow[dl, "g_1"] \\
 & S_1 &
\end{tikzcd}$$

to a map

$$g_{0,*}f_0^! \to g_{1,*}f_1^!.$$

Our expression of the Bauer–Furuta invariant $r_*f^! \to q_*$ is nothing but the value under this $(\infty, 2)$-functoriality of the following 2-correspondence.

$$\begin{tikzcd}
 & C & \\
C' \arrow[ur, "f"] \arrow[dr, "r"'] \arrow[r, equal] & C' \arrow[r, "f"] & C \arrow[ul, "\mathrm{id}"'] \arrow[dl, "q"] \\
 & \mathrm{pt} &
\end{tikzcd}$$

1.3. **Organization of the paper.** The construction of the six-functor formalism for sheaves of spectra on general topological spaces will be given in Section 3. We would like to warn the reader that this paper mainly focuses on the *non-equivariant* version of the Bauer–Furuta construction, which is the content of Section 2. The group equivariance, however, plays a crucial role in the applications of the Bauer–Furuta invariants to low-dimensional topology. To obtain such a genuine equivariant refinement in a parallel way, we need some genuine equivariant six-functor formalism, which we hope to exist but is not fully developed yet. In Section A, we will outline a portion of such a genuine equivariant six-functor formalism, and we at least define the genuine equivariant version of the Bauer–Furuta construction. Detailed studies on the properties of the genuine equivariant Bauer–Furuta construction, as defined so, will be explored in a subsequent paper.

1.4. **Further directions.** In low-dimensional topology, gauge theory has produced a variety of invariants of differentiable structures on manifolds by blending techniques from differential geometry, functional analysis, and algebraic topology. Since these invariants take values in homological algebra or stable homotopy theoretic objects, one should expect them to be governed by algebraic topology—or perhaps by some new form of "motives." However, a completely satisfactory topological theory for these invariants is still lacking. In this paper, we investigate one possible answer to the question of how far pure algebraic topology can explain a gauge theoretic invariant. While the primary focus of this paper is to establish a new construction of the Bauer–Furuta invariant within a six-functor framework, we expect that our methods have broader applications in gauge-theoretic contexts. A central to our results is that (nonlinear) Fredholm maps naturally fit into our six-functor formalism: they admit exceptional shriek functors and Borel–Moore-type theories are well-defined along them. This observation opens the door to a sheaf-theoretic perspective on gauge theory that has been surprisingly unexplored. Indeed, while cohomological ideas lie at the heart of many constructions in gauge theory, these are rarely approached via sheaves of spectra. For example, the transfer map mentioned in the main theorem should be viewed as a spectrum-level generalization of the integration along fibers—echoing the traditional viewpoint of the Seiberg–Witten invariant as an integration of a certain Euler class.

We hope this perspective will serve as a starting point for sheaf-theoretic constructions of other Floer-type invariants and their families. One notable aspect of our approach is that it relies only on the topological structure of the nonlinear map defined by a gauge-theoretic equation. This is in sharp contrast with traditional constructions, which often involve additional functional analysis input beyond topology. Our results thus indicate that a purely topological theory is possible in many cases.

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1.5. **Notations.** *Categories* mean $(\infty, 1)$-categories. Cat, An, Pr^{L}, $\widehat{\mathsf{Cat}}$ and Sp denote the category of small categories, the category of small $(\infty, 0)$-categories, the category of presentable categories (with morphisms the left adjoint functors), the category of locally small categories, and the category of spectra, respectively. Pr^{L} is equipped with the symmetric monoidal structure which gives the tensor product of presentable categories. In a *presentably symmetric monoidal category*, i.e. an object of $\mathsf{CAlg}(\mathsf{Pr}^{\mathrm{L}})$, we use $\otimes$, $\mathbf{1}$, and $\underline{\mathrm{Hom}}$ to denote the tensor product, the monoidal unit and the internal hom, respectively. For example, Sp is equipped with the presentably symmetric monoidal structure, which is an idempotent algebra in Pr^{L} corresponding to the localization onto the full subcategory $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{st}}$ of *stable presentable categories*. The monoidal unit for Sp, the *sphere spectrum*, is also denoted by $\mathbb{S}$.

2. On the Bauer–Furuta construction

Convention 2.0.1. In this section, all topological spaces are assumed to be Hausdorff for simplicity[6], so that universally closed maps are proper and in particular locally proper (Theorem 3.1.1).

Sheaves will always take values in Sp, the category of spectra. For a topological space Y, $\mathsf{Sh}(Y)$ will denote the category of sheaves of spectra $\mathsf{Sh}(Y;\mathsf{Sp})$ defined in [Lur09, 6.2.2.6, 6.3.5.15], see Theorem 3.0.1 also. We use the six operations $f^*, f_*, f_!, f^!, \otimes, \underline{\mathrm{Hom}}$ on those sheaves, which are summarized in Section 3 (especially in Theorem 3.4.1). A base topological space will often be denoted by S.

2.1. **From Borel–Moore functoriality to Bauer–Furuta.**

Construction 2.1.1. Let $f\colon L \to Y$ be a proper map between topological spaces over a base S.

$$\begin{tikzcd} L \arrow[rr, "f"] \arrow[dr, "r"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd} \tag{2.1}$$

Since f is proper, we have the functor $f^!$ right adjoint to f_*. Thus, we have the map (in $\mathsf{Sh}(S)$) of the following form.

$$\mathrm{BF}_f\colon r_*f^!(\mathbf{1}) = q_*f_*f^!(\mathbf{1}) \xrightarrow{q_*(\mathrm{counit})} q_*(\mathbf{1})$$

This will be our definition of the *Bauer–Furuta map*.

It is nothing but the usual proper pushforward map $\mathrm{H}^{\mathrm{BM}}_{\bullet}(L/Y;\mathbb{S}) \to \mathrm{H}^{\mathrm{BM}}_{\bullet}(Y/Y;\mathbb{S}) = \mathbf{1}_{\mathsf{Sh}(Y)}$ on the Borel–Moore homology, applied to the functor $q_*\colon \mathsf{Sh}(Y) \to \mathsf{Sh}(S)$. An immediate observation is that this construction is pullback-stable:

Remark 2.1.2. Consider the following diagram of topological spaces in which f is proper.

$$\begin{tikzcd} L' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, "g"] \arrow[dr, "q'"] & \\ L \arrow[r, "f"'] & Y \arrow[r, "q"'] & S \end{tikzcd}$$

Then we have a commutative square of the following form.

$$\begin{tikzcd} f_*f^! \arrow[r, "\mathrm{counit}"] \arrow[d, "f_*f^!(\mathrm{unit})"'] & \mathrm{id} \arrow[d, "\mathrm{unit}"] \\ f_*f^!g_*g^* \arrow[r] & g_*g^* \end{tikzcd}$$

Here the bottom horizontal map is the composite of the proper basechange isomorphism and the counit map as follows.

$$f_*f^!g_*g^* \xleftarrow{\sim} f_*g'_*f'^!g^* = g_*f'_*f'^!g^* \xrightarrow{g_*(\mathrm{counit})} g_*g^*$$

Here the first map is the isomorphism obtained by taking the right adjoints of the (proper) basechange isomorphism $g^*f_* \simeq f'_*g'^*$. In particular, we have a diagram

$$\begin{tikzcd} \mathrm{BF}_f\colon q_*f_*f^!(\mathbf{1}) \arrow[r] \arrow[d] & q_*\mathbf{1} \arrow[d] \\ \mathrm{BF}_{f'}\colon q'_*f'_*f'^!(\mathbf{1}) \arrow[r] & q'_*\mathbf{1}. \end{tikzcd} \tag{2.2}$$

Note that if g is contractible (Theorem 3.5.3), then both vertical maps are isomorphisms, so that we identify BF_f with $\mathrm{BF}_{f'}$.

The argument here is functorial at least up to homotopy: If we further basechange along $h\colon Y'' \to Y'$, the commutative squares given above compose and form a rectangle.

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[6] Another option is to assume that every map we will care is separated.

The rest of this section is devoted to explaining how this map BF_f subsumes the construction of [BF04]. We here record the abstraction of basic properties satisfied by the Seiberg–Witten map (1.3).

Remark 2.1.3. We will be interested in the following situation:

$$\begin{tikzcd} L \arrow[rr, "f"] \arrow[dr, "r"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

f is a C^1-differentiable (Theorem C.2.1) and proper map between Banach vector bundles (Theorem C.1.1) over S such that the (vertical) differential $df\colon L \times_S L \to L \times_S Y$ is a linear Fredholm map at every point. In practice, it is often the case that any differentials at two distinct points only differ by a compact linear operator.

Example 2.1.4 ([BF04])**.** Let
$$\begin{tikzcd} E' \arrow[rr, "f"] \arrow[dr, "r"'] & & E \arrow[dl, "q"] \\ & S & \end{tikzcd}$$
be a map between Hilbert vector bundles which is given as $f = l + c$ for l a linear Fredholm map and c a compact map (in the sense that c maps a bounded disk bundle of E'_K to a relatively compact subset in E_K for each compact subset $K \subset S$). Assume that on each fiber, the map $f_x\colon E'_x \to E_x$ has bounded preimages of bounded sets.[7] Then [BF04, Lemma 2.2] says that each map $f_x\colon E'_x \to E_x$ on fibers is proper. In the same way, it follows that if S is compactly generated, then $f\colon E' \to E$ is a proper map. See Theorem B.2.2 for a discussion.

Remark 2.1.5. A typical example of a base S that will be considered in the Bauer–Furuta construction is either the torus $H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$, the space of riemannian metrics and connections, or the space of spin^c-structures. We will remark on this point in Section B. In particular, we will need a locally non-compact base space S.

We have a fundamental decomposition for Fredholm maps as follows.

Lemma 2.1.6. *Let $f\colon L \to Y$ be a C^1-differentiable map between Banach vector bundles over S with Fredholm differentials. Then locally on the source, the map f factors as a composite of the following form.*

$$\begin{tikzcd} L \arrow[rrrd, "f"'] & \supset & U \arrow[r, "e"] & U \times_Y T \arrow[d, "p"] \\ & & & Y \end{tikzcd}$$

Here T is a finite-rank vector bundle over an open subspace of Y, e is the zero-section, and p is cohomologically smooth.

Proof. Given a point $v \in L_x \subset L$ on the source, let T_0 denote a closed linear subspace of Y_x complementary to $\mathrm{im}(d_v f)$, which exists since the derivative is Fredholm. Choose a neighborhood S_0 of x and trivializations $L\mid_{S_0} \cong S_0 \times L_x$ and $Y\mid_{S_0} \cong S_0 \times Y_x$. Define the map $p\colon S_0 \times L_x \times T_0 \to S_0 \times Y_x$ by the formula

$$p(x', v', w) = (x', f(v') + w),$$

so that it is of C^1-class and the (vertical) derivative at the point $(x, v, 0)$ is surjective and has kernel isomorphic to the kernel of $d_v f$. Therefore, by virtue of Theorem C.2.3, we find a vector space K_0 isomorphic to $\ker(d_v f)$ and open neighborhoods V of $f(v)$ in Y, E of 0 in T_0 and U of v in L such that the map $p\colon S_0 \times_S L \to Y$ is locally of the following form.

$$U \times E \quad \cong \quad V \times K_0 \xrightarrow{\mathrm{pr}} V$$

Since K_0 is finite-dimensional, it follows that p is cohomologically smooth (Theorem 3.6.5). The desired factorization is obvious from the definition of p. ☐

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[7] This is the case for the actual Seiberg–Witten map (1.3).

We have the following corollaries.

Corollary 2.1.7. *Let f be as in Theorem 2.1.6. Then f is locally proper (Theorem 3.1.1). In particular, the functor $f^!\colon \mathsf{Sh}(Y) \to \mathsf{Sh}(L)$ is well-defined (Theorem 3.4.1).*

Proof. The property of being locally proper can be checked locally on the source and the target. By Theorem 2.1.6, f is locally a composition of e, which is proper, and p, which is locally proper. ☐

Before stating the next corollary, let us recall the construction of Thom spectra.

Construction 2.1.8. For a finite-rank vector bundle $q\colon V \to S$, define the *Thom spectrum sheaf* $\mathrm{Th}(V)$ to be $q_*q^!(\mathbf{1}) \in \mathsf{Sh}(S)$. It is also isomorphic to $q_\sharp \operatorname{cof}(j_!(\mathbf{1}) \to \mathbf{1})$ where $j\colon V^\times \to V$ is the complement of the zero-section, which is informally interpreted as the homotopy type of the cofiber $V/V^\times$.

This construction promotes to a symmetric monoidal functor defined on the groupoid of finite-rank vector bundles

$$\mathrm{Th}\colon (\mathrm{Vect}(S)^\simeq, \oplus) \to (\mathsf{Sh}(S), \otimes)$$

by employing the (lax symmetric monoidal) functoriality of the six-functor formalism (Theorem 3.4.1) sending a correspondence $S \leftarrow V \to S$ to a linear functor $q_!q^*$ which is left adjoint to $q_*q^!$. Since it also takes values in $\otimes$-invertible objects, the functor Th extends to a symmetric monoidal functor defined on virtual vector bundles.

Next, consider a linear Fredholm map $l\colon \mathcal{H}' \to \mathcal{H}$ between Banach vector bundles over S and let $r\colon \mathcal{H}' \to S$ denote the projection. We let $r_*l^!(\mathbf{1})$ be suggestively denoted by $\mathrm{Th}(\mathrm{ind}(l))$, which we refer to as the *Thom spectrum sheaf of families index.*

Remark 2.1.9. We here remark and justify our terminology. If S is at least compact (Hausdorff), then the functor

$$\mathrm{Th}(\mathrm{ind}(-))\colon \pi_0(\mathrm{Fred}(S)^\simeq) \to \mathsf{Sh}(S)$$

does factor through the Atiyah–Singer families index, which takes values in $\pi_0 KO(S)$. To see this, we observe that if S is compact, then there exists a finite-rank subbundle W of $\mathcal{H}$ over S that is transverse to l in the sense that the image of l and W span $\mathcal{H}$. Thus, $V := W \times_{\mathcal{H}} \mathcal{H}'$ is a finite-rank subbundle of $\mathcal{H}'$ due to transversality and being Fredholm. Then l factors as $l = pe$ as in Theorem 2.1.6 where $e\colon \mathcal{H}' \to W \times_S \mathcal{H}'$ is the zero-section and p has kernel V. By Theorem 3.6.7, $l^!(\mathbf{1}) = e^!p^!(\mathbf{1})$ is isomorphic to $e^!(\mathbf{1}) \otimes e^*p^!(\mathbf{1})$, which in turn is identified with $r^*\mathrm{Th}(-W) \otimes \mathrm{Th}(e^*T_p) \simeq r^*\mathrm{Th}(V - W)$. Therefore, $r_*l^!(\mathbf{1})$ is isomorphic to the Thom spectrum sheaf of the virtual vector bundle $V - W$, which is by definition the Atiyah–Singer index $\mathrm{ind}(l) \in \pi_0 KO(S)$.

Corollary 2.1.10 (Descent to local systems)**.** *Let f be as in Theorem 2.1.6. Then $f^!(\mathbf{1})$ is locally constant and $\otimes$-invertible. Therefore, the map BF_f lies over the full subcategory of locally constant sheaves on S.*

Proof. Taking a local decomposition $fj_U = pe$ provided by Theorem 2.1.6 and argue as in Theorem 2.1.9, the sheaf $j_U^*f^!(\mathbf{1})$ is locally of the form $\mathrm{Th}(-N_e) \otimes \mathrm{Th}(e^*T_p)$, which is the Thom spectrum sheaf of a virtual vector bundle and thus a locally constant and invertible sheaf.

Then by Theorem 3.5.7, $r_*f^!(\mathbf{1})$ is equivalent to $x^*f^!(\mathbf{1})$ for, say, the zero section $x\colon S \to L$. In particular, it is also locally constant. ☐

The preceding corollary may be pleasant for the reason that over a homotopically well-behaved topological space S, locally constant sheaves correspond to local systems, also known as *parameterized spectra* due to Theorem 3.5.8.

Construction 2.1.11. Let f be as in Theorem 2.1.6 and assume further for simplicity that any differentials at two distinct points differ only by a compact linear operator. Let df denote the differential $L \times_S L \to L \times_S Y$, which is a linear map between Banach vector bundles $p\colon L \times_S L \to L$ and $L \times_S Y \to L$. We construct an isomorphism $f^!(\mathbf{1}) \simeq p_*(df)^!(\mathbf{1})$ as follows. Define a map

$$\phi\colon [0,1] \times L \times_S L \to [0,1] \times L \times_S Y$$

by the formula $\phi(t,x,v) = (t,(1-t)f(v)+t(d_xf)v)$. By our assumption on f, ϕ has Fredholm differentials at each point. We repeat the above arguments for ϕ to conclude that $\phi^!(\mathbf{1})$ is locally constant (and hence constant along the $[0,1]$-direction by Theorem 3.5.6), so that

$$(df)^!(\mathbf{1}) \simeq i_1^*\phi^!(\mathbf{1}) \simeq i_0^*\phi^!(\mathbf{1}) \simeq p^*f^!(\mathbf{1})$$

Here we used Theorem 3.6.8 in the first and the third identifications.

Corollary 2.1.12 (Linearization hypothesis)**.** *Let f be as in Theorem 2.1.11. Then $f^!(\mathbf{1})$ can be identified with* $\mathrm{Th}(\mathrm{ind}(df))$ *the Thom spectrum sheaf of the index of the Fredholm operator df. Taking a section $x\colon S \to L$ of r, the $r_*f^!(\mathbf{1})$ can be identified with* $\mathrm{Th}(\mathrm{ind}(d_xf))$*. The latter identification works as well if only f is locally proper and C^1-differentiable near the section $x\colon S \to L$ with Fredholm differentials.*

Proof. The fist claim follows from the deformation argument given in Theorem 2.1.11. The second claim follows since $r_*f^!(\mathbf{1}) \simeq x^*f^!(\mathbf{1})$ again by Theorem 3.5.7. ☐

Thus far, by choosing a section $x\colon S \to L$, we have observed that our Bauer–Furuta map can be seen as a map of the form

$$\mathrm{BF}_f\colon \mathbb{S}^{\mathrm{ind}(d_xf)} \to \mathbb{S}$$

in $\mathrm{LocSys}(S;\mathsf{Sp})$ (for a nice base S). Later, we will provide a way to compare our construction to the classical Bauer–Furuta construction given by the finite-dimensional approximations as in (1.6). This will be discussed in Theorem 2.3.1. It is important to note that the finite-dimensionally approximated map (1.4) itself may not be a proper map. So we would like to provide a variant of our Bauer–Furuta map which does not assume the map to be proper.

2.2. Some purity results for BF.

Here we are going to explore how a smaller subspace of L will suffice to construct the Bauer–Furuta map of Theorem 2.1.1.

Notation 2.2.1. Let a map f in the following be locally proper and fix a section $s\colon S \to Y$ of q. Consider the following pullback of topological spaces over S.

$$\begin{tikzcd}
M \arrow[r, "\pi"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "s"'] \arrow[dd, bend left, equal] \\
L \arrow[r, "f"] \arrow[dr, "r"'] & Y \arrow[d, "q"'] \\
& S
\end{tikzcd} \tag{2.3}$$

The topological space M can be thought of as the moduli space, i.e., an affine cover of the moduli stack[8], of solutions to the equation $f = s$.

Remark 2.2.2. Since all spaces are assumed to be Hausdorff (Theorem 2.0.1), the sections s and i are closed immersions.

In the case when the (affine cover of the) moduli (stack) is proper, we have a variant of Theorem 2.1.1.

Construction 2.2.3. Assume that the map π in (2.3) is proper. Construct another map as follows.

$$r_*f^!(\mathbf{1}) \xrightarrow{r_*f^!(\mathrm{unit})} r_*f^!s_*(\mathbf{1}) \xrightarrow{\sim} r_*i_*\pi^!(\mathbf{1}) = \pi_*\pi^!(\mathbf{1}) \xrightarrow{\mathrm{counit}} \mathbf{1} \xrightarrow{\mathrm{unit}} q_*(\mathbf{1}) \tag{2.4}$$

Here the second map is the isomorphism $f^!s_* \simeq i_*\pi^!$ obtained by the basechange isomorphism $s^*f_! \simeq \pi_!i^*$.

If q is contractible, then the unit map $\mathbf{1} \to q_*\mathbf{1}$ is inverse to the map $\mathbf{1} \to q_*\mathbf{1}$ (by Theorem 3.5.4). Thus, if moreover f is proper, then the map BF_f and the map (2.4) coincide by Theorem 2.1.2.

---

[8] The term *moduli stack* refers to some stacky quotient by the gauge group action.

Corollary 2.2.4. *If in the diagram (2.3) f is proper and q is contractible, then the map BF_f in Theorem 2.1.1 and the map (2.4) coincide.*

Remark 2.2.5. Assuming that π is proper, we have yet another variant

$$r_*f^!(\mathbf{1}) \xrightarrow{r_*(\mathrm{unit})} r_*i_*i^*f^!(\mathbf{1}) = \pi_*i^*f^!(\mathbf{1}) \xleftarrow{\sim} s^*f_!f^!(\mathbf{1}) \xrightarrow{s^*(\mathrm{counit})} s^*(\mathbf{1}) = \mathbf{1} \xrightarrow{\mathrm{unit}} q_*(\mathbf{1})$$

where the second map is the basechange isomorphism $\pi_*i^* \simeq s^*f_!$. By some simple diagram chasing, one can show that the map coincides with (2.4), also with Theorem 2.1.1 under the same assumption as above.

Example 2.2.6. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $S = \mathrm{pt}$, and assume that f have compact zeros $f^{-1}(0) = M$. Letting s be the map $\{0\} \to \mathbb{R}^m$, Construction 2.2.3 also produces a map $r_*f^! \to q_*$, and hence a map

$$\mathbb{S}^n = r_*f^!q^!(\mathbf{1}) \longrightarrow q_*q^!(\mathbf{1}) = \mathbb{S}^m,$$

which is the suspension of (2.4) by m. This map is equivalent to (the stabilization of) the following map on the homotopy cofibers.

$$S^n \longrightarrow \mathbb{R}^n/(\mathbb{R}^n - M) \xrightarrow{f} \mathbb{R}^m/(\mathbb{R}^m - \{0\}) = S^m$$

Here the first map collapses outside M. This reduces the Bauer–Furuta construction to Construction 2.2.3.

The map (2.4) can be interpreted as the restriction map onto the Borel–Moore $\mathbb{S}$-homology of the moduli M, followed by the proper pushforward, i.e., the integration map along π. This is a geometric interpretation of our Bauer–Furuta map.

In the Seiberg–Witten map, the choice of a section s can be thought of a perturbation of the moduli space M; generically s can be transverse to the map f. The propositions above thus imply the following.

Corollary 2.2.7 (Independence of the perturbation)**.** *Assume that q is contractible. If f is proper, then the map (2.4) is independent of the choice of a section s.*

We may, therefore, abuse notation and write BF_f for the map $r_*f^!(\mathbf{1}) \to \mathbf{1}$ as in (2.4) whenever we are given a section s with f being only locally proper and π being proper. As we have just observed, it does not depend on the choice of a section when f is proper.

Another obvious corollary is that in the construction (2.4), only the values of f near the moduli M is crucial. This is why we consider these comparison results as purity phenomena. We record this observation as follows.

Corollary 2.2.8 (A purity for the Bauer–Furuta map)**.** *Assume, in the diagram (2.3), that π is proper. Take an open subspace $j\colon N \subset L$ and let f_N denote the restriction $N \subset L \xrightarrow{f} Y$. Suppose that the map $N \cap M \hookrightarrow M \xrightarrow{\pi} S$ is proper. Then we have a (commutative) diagram of the following form.*

$$\begin{tikzcd}
r_*f^!(\mathbf{1}) \arrow[rr, "\mathrm{BF}_f"] \arrow[d] & & \mathbf{1} \\
r_*j_*j^*f^!(\mathbf{1}) \arrow[r, phantom, "="] & r_*j_*f_N^!(\mathbf{1}) \arrow[ur, "\mathrm{BF}_{f_N}"'] &
\end{tikzcd}$$

If, moreover, $f^!(\mathbf{1})$ is constant along r and j is a homotopy equivalence over the base, then the left vertical map is an isomorphism and thus we conclude that $\mathrm{BF}_f \simeq \mathrm{BF}_{f_N}$.

2.3. **Comparison to finite-dimensional approximations.** We end this section by proving a promised comparison result to the classical Bauer–Furuta construction given by the finite-dimensional approximations.

Let us focus on the Seiberg–Witten map, so that we assume that f is as in Theorem 2.1.4. We further assume for simplicity that our base S is a point. So L and Y are simply Banach vector spaces.

For any topological space B, we let p_B denote the unique map $B \to *$ onto a point. For any map $h\colon B \to S$ and any subspace $C \subset B$, we write h_C for the restriction of h to C.

The proof given below is very similar to that of [BF04, Lemma 2.3 (3)].

Theorem 2.3.1 (Finite-dimensional approximation)**.** *Let $f\colon L \to Y$ be a proper and C^1-differentiable map between Banach vector spaces, which we assume to be of the form $f = l + c$ where l is a linear Fredholm map and the differentials of c are compact operators.*

Assume that there exists a finite-dimensional linear subspace $\iota\colon W \subset Y$ that spans Y together with $\mathrm{im}(l)$ and admits a splitting $\mathrm{pr}_W\colon Y \to W$. Let W' denote $l^{-1}(W)$, which is also finite-dimensional.

$$\begin{tikzcd}
W' \arrow[r, "l'"] \arrow[d, "\iota'"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "\iota"] \\
L \arrow[r, "l"] & Y
\end{tikzcd}$$

Consider the composite $g\colon W' \hookrightarrow L \xrightarrow{f} Y \xrightarrow{\mathrm{pr}_W} W$. Since $M := f^{-1}(0)$ is compact, it is bounded in L. Let us assume further that there exists a bounded open subspace $j\colon N \hookrightarrow L$ containing $f^{-1}(0)$ such that $N \cap f^{-1}\mathrm{pr}_W^{-1}(0)$ is compact.[9]

$$\begin{tikzcd}
g\colon W' \arrow[r, "\iota'"] & L \arrow[r, "f"] & Y \arrow[r, "\mathrm{pr}_W"] & W \\
N' \arrow[u, "j'"] \arrow[r, "\iota''"'] \arrow[ur, phantom, "\urcorner", very near start] & N \arrow[u, "j"']
\end{tikzcd}$$

In particular, the map $g_{N'}$ has compact zero-set and the construction (2.4) makes sense.

Then the map $\mathrm{BF}_f\colon \mathbb{S}^{\mathrm{ind}(l)} \simeq p_{L*}f^!(\mathbf{1}) \to \mathbb{S}$ can be identified with the following composite map, which can be thought of as the classical Bauer–Furuta map obtained by the finite-dimensional approximation argument:

$$\mathrm{BF}_f^{\mathrm{apprx}}\colon p_{W'*}g^!(\mathbf{1}) \to p_{N'*}j'^*g^!(\mathbf{1}) = p_{N'*}g^!_{N'}(\mathbf{1}) \xrightarrow{\mathrm{BF}_{g_{N'}}} \mathbb{S}$$

Proof. Let $\mathcal{H} = \ker(\mathrm{pr}_W)$. We first choose an isomorphism $L \cong \mathcal{H} \times W'$ such that the map $\mathcal{H} \times W' \cong L \xrightarrow{l} Y = \mathcal{H} \times W$ is of the form $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$, which exists since l is Fredholm. Define a map g' as follows.

$$\begin{tikzcd}
\mathrm{pr}_{W'}^{-1}(N') =: N'' \arrow[d, dash, "\cong"] & g'\colon L \arrow[r] \arrow[d, dash, "\cong"] & Y \arrow[d, equal] \\
\mathcal{H} \times N' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{H} \times W' \arrow[r] \arrow[d, "\mathrm{pr}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{H} \times W \arrow[d, "\mathrm{pr}_W"] \\
N' \arrow[r, phantom, "\subset"] & W' \arrow[r, "g"] & W
\end{tikzcd}$$

In other words, $g' = l + \mathrm{pr}_W \circ c \circ \mathrm{pr}_{W'}$. The map g' also has differentials of the form $l +$ *compact* as well as f, and $g'_{N''}$ has compact zero-set as well as $g_{N'}$. By the pullback-stability (Theorem 2.1.2) we conclude that $\mathrm{BF}_{g'_{N''}} \simeq \mathrm{BF}_{g_{N'}}$.

Construct the map over the base space $[0,1]$

$$\begin{tikzcd}
\psi\colon L \times [0,1] \arrow[rr] \arrow[dr] & & Y \times [0,1] \arrow[dl] \\
& {[0,1]} &
\end{tikzcd}$$

by the formula $\psi(-,t) = ((1-t)f + t(l + \mathrm{pr}_W \circ c), t)$. The vertical differentials of ψ are of the form $l +$ *compact* hence Fredholm, and the zeros of $\psi_{N \times [0,1]}$ are contained in $N \cap f^{-1}\mathrm{pr}_W^{-1}(0)$ hence compact. Similarly, we define another 1-parameter family ϕ by the formula $\phi(-,t) = (l + \mathrm{pr}_W \circ c \circ ((1-t)\mathrm{id} + t\mathrm{pr}_{W'}), t)$. It has Fredholm differentials, and the zeros of $\phi_{N \times [0,1]}$ are contained in the compact $N \cap f^{-1}\mathrm{pr}_W^{-1}(0)$. We now execute the construction (2.4) to the

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[9] Note that all the assumptions are satisfied by the Seiberg–Witten map by [BF04, 2.2, 2.3, 3.1].

maps ψ and ϕ and then invoke locally constancy (Theorem 2.1.10) and homotopy invariance (Theorem 3.5.6) to conclude that $\mathrm{BF}_{f_N} \simeq \mathrm{BF}_{g'_N}$.

Finally, we have the purity triangles (Theorem 2.2.8) of the form $\mathrm{BF}_f \to \mathrm{BF}_{f_N}$ and $\mathrm{BF}_{g'_N} \to \mathrm{BF}_{g'_{N\cap N''}} \leftarrow \mathrm{BF}_{g'_{N''}}$, which we consider as the maps in the slice category $\mathsf{Sp}_{/\mathbb{S}}$.

Combining all together, we get a zigzag connecting the desired maps.

$$\begin{tikzcd}
\mathbb{S}^{\mathrm{ind}(l)} \simeq \bullet \arrow[r, "\mathrm{BF}_f"] \arrow[d] & \mathbb{S} \\
{} & \\
\mathbb{S}^{\mathrm{ind}(l)} \simeq \bullet \arrow[u] \arrow[uur, "\mathrm{BF}^{\mathrm{apprx}}"'] &
\end{tikzcd}$$

We also observe that identifications of the left hand sides with $\mathbb{S}^{\mathrm{ind}(l)}$ can be given as in Theorem 2.1.11, which commute with the vertical zigzag that we have constructed so far. Therefore, we have completed constructing a homotopy $\mathrm{BF}_f \simeq \mathrm{BF}_f^{\mathrm{apprx}}$. ☐

We do not think that this theorem will serve a substantial role in our formalism, as we believe that all the basic behaviors satisfied by the Bauer–Furuta invariants can be proved in a simpler way within our definition. Nonetheless, the theorem serves at least as a justification that our construction subsumes the constructions of [Fur01] and [BF04].

3. THE SIX-FUNCTOR FORMALISM FOR SHEAVES ON TOPOLOGICAL SPACES

This section is aimed to recall the six-functor formalism for sheaves of spectra on topological spaces. It serves as a spectral refinement of the work of [SS16], where the functors $f_! \dashv f^!$ are defined for separated locally proper maps. One can compare to the paper [Vol23], which provides the six-functor formalism for locally compact Hausdorff spaces. Since we need the shriek functors along Fredholm maps between locally non-compact spaces (such as infinite-dimensional Banach spaces), we record the construction of that six-functor formalism, using the arguments in [Man22, A.5] pioneered by Liu–Zheng. We would like to note that it may also be possible to provide such a six-functor formalism by developing the theory of *locally rigid algebras* [Ram24] in $\mathsf{Mod}(\mathsf{Pr}^{\mathrm{L}}_{\mathrm{st}})$ since separated locally proper maps of topological spaces are typical examples. We, however, do not pursue that perspective in this paper.

Let Top denote the $(1,1)$-category of topological spaces whereas Top_∞ denote the category of ∞-topoi and geometric morphisms between them. We write $\mathsf{Top}_\infty^{\mathrm{L,lex}}$ for the subcategory of Pr^{L} equivalent to $\mathsf{Top}_\infty^{\mathrm{op}}$. Recall the functor

$$(\mathsf{Sh}(-;\mathsf{An}), (-)^*) \colon \mathsf{Top}^{\mathrm{op}} \to \mathsf{Pr}^{\mathrm{L}}$$

from [Lur09, Ch. 6] given as the following composite.

$$\mathsf{Top}^{\mathrm{op}} \xrightarrow{\mathrm{Open}(-)} \mathsf{Locale}^{\mathrm{L,lex}} \simeq \mathsf{Top}^{\mathrm{L,lex}}_{\infty,0\text{-loc}} \hookrightarrow \mathsf{Top}^{\mathrm{L,lex}}_\infty \xhookrightarrow{\text{forgetful}} \mathsf{Pr}^{\mathrm{L}}$$

It takes a topological space Y to the category of sheaves in An, which is uniquely characterized by the following universal property

$$\mathsf{Fun}^{\mathrm{L}}(\mathsf{Sh}(Y;\mathsf{An}), \mathcal{C}) \xrightarrow[\sim]{} \mathsf{Fun}^{\mathrm{eff.epi}}(\mathrm{Open}(Y), \mathcal{C})$$

for $\mathcal{C}$ a presentable category. (In fact this characterization proves the above functoriality.) Here $\mathsf{Fun}^{\bullet}(-,-)$ is the category of functors preserving colimit diagrams specified by $\bullet$.

Notation 3.0.1. For a topological space Y, let $\mathsf{Sh}(Y)$ denote the category $\mathsf{Sh}(Y;\mathsf{Sp})$ of sheaves of spectra, which is equivalent to $\mathsf{Fun}^{\mathrm{R}}(\mathsf{Sh}(Y;\mathsf{An})^{\mathrm{op}}, \mathsf{Sp})$ and to $\mathsf{Sh}(Y;\mathsf{An}) \otimes \mathsf{Sp}$. Thus, we have the $*$-functoriality:

$$\mathsf{Sh}(-) \quad : \quad \mathsf{Top}^{\mathrm{op}} \longrightarrow \mathsf{CAlg}(\mathsf{Pr}^{\mathrm{L}}_{\mathrm{st}})$$

$$\begin{tikzcd}
Y \arrow[d, "f"'] & \mapsto & \mathsf{Sh}(Y) \\
X & & \mathsf{Sh}(X) \arrow[u, "f^*"']
\end{tikzcd}$$

The right adjoint to the functor f^* is denoted by f_*.

In what follows, we will freely use the following fact.

Remark 3.0.2. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be all presentable. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a colimit-preserving functor.

If f admits a fully faithful right adjoint f^{R}, then the functor $(f \otimes \mathrm{id}_{\mathcal{E}})^{\mathrm{R}}$ right adjoint to the functor $f \otimes \mathrm{id}_{\mathcal{E}}\colon \mathcal{C} \otimes \mathcal{E} \to \mathcal{D} \otimes \mathcal{E}$ is fully faithful.

If the functor f^{R} right adjoint to f admits a further right adjoint f^{RR}, then the functor $(f \otimes \mathrm{id})^{\mathrm{R}}$ is isomorphic to $f^{\mathrm{R}} \otimes \mathrm{id}_{\mathcal{E}}$. Combined with the first remark, every functor $f\colon \mathcal{C} \to \mathcal{D}$ that admits a colimit-preserving fully faithful right adjoint f^{R} is tensored up to an adjoint triple $f \otimes \mathrm{id}_{\mathcal{E}} \dashv f^{\mathrm{R}} \otimes \mathrm{id}_{\mathcal{E}} \dashv (f^{\mathrm{R}} \otimes \mathrm{id}_{\mathcal{E}})^{\mathrm{R}}$ with $f^{\mathrm{R}} \otimes \mathrm{id}_{\mathcal{E}}$ fully faithful.

The first fact follows from how a tensor product can be constructed in terms of presentations. The second fact follows from the observations that such an f is an internal left adjoint in Pr^{L} and that $(\mathsf{Pr}^{\mathrm{L}}, \otimes)$ has a structure of a symmetric monoidal 2-category.

3.1. Locally proper maps.

Convention 3.1.1. We here fix some basic terminology regarding compactness.

1. We say a map between topological spaces is *proper* if it is separated and universally closed. For example, a map onto a singleton is proper if and only if the topological space is compact Hausdorff.
2. We say a map f is *locally universally closed* if it is locally proper in the sense of [SS16, Definition 2.3]. That is, if for any (open) neighborhood V of a point y on the source there exist a (open) neighborhood U of $f(y)$ in the target and a neighborhood $N \subset f^{-1}(U) \cap V$ of y such that f restricts to a universally closed map $N \to U$. For example, a map onto a singleton is locally universally closed if and only if the topological space is locally compact.
3. We define a *locally proper map* in the same way: For any neighborhood V of a point y on the source, there exist neighborhoods U of $f(y)$ and $N \subset f^{-1}(U) \cap V$ of y such that the restriction $f\colon N \to U$ is proper.

   We remark that separated and locally universally closed maps are automatically separated locally proper.
4. An *immersion* is a map that induces a homeomorphism onto the image which is locally closed in the target. In other words, an immersion is a locally proper embedding.

The following well-known lemma, the existence of a compactification relative to a base, will provide later (Theorem 3.4.1) well-defined shriek functors for such maps. One can refer to [Jam89, II. 8], and we do not claim any originality of the following argument.

Lemma 3.1.2. *A map $f\colon Y \to S$ is separated locally proper if and only if it is of the form $f = pj$ where $j\colon Y \to X$ is an open embedding and $p\colon X \to S$ is a proper map.*

Proof. Since both p and j are separated locally proper and since the collection of separated locally proper maps are closed under compositions, any map of the form $f = pj$ is separated locally proper.

Conversely, we wish to construct an explicit fiberwise one-point compactification Y^{+S} of a separated locally proper map f. Define a topology on the underlying set $Y \sqcup S$ as follows.

1. For each open $V \subset Y$, the subset $V \sqcup \varnothing \subset Y \sqcup S$ is open.
2. For each $x = f(y) \in f(Y)$ and for each neighborhoods $U \ni x$ and $N \ni y$ with $f\colon N \to U$ proper, the subset $\bigl(f^{-1}(U) - N\bigr) \sqcup U$ is a neighborhood of $x \in \varnothing \sqcup S \subset Y \sqcup S$.
3. For each point $x \notin f(Y)$ of S and for each neighborhood U of x, the subset $f^{-1}(U) \sqcup U$ is a neighborhood of $x \in \varnothing \sqcup S \subset Y \sqcup S$.
4. Define the topological space Y^{+S} to be the final topology on $Y \sqcup S$ satisfying all the above conditions. Each of the conditions (2) and (3) forms a neighborhood basis of that point x.

Then the obvious maps $j\colon Y \to Y^{+S}$ and $p\colon Y^{+S} \to S$ are continuous, j is open, p is separated and p has (quasi)compact fibers. We need to prove that p is a closed map. In other words, we wish to show that for any point $x \in S$, every neighborhood of $p^{-1}(x)$ contains a subset of the form $p^{-1}(U)$ for some neighborhood U of x.

First, for a point $x \notin f(Y)$ of S, any neighborhood of $p^{-1}(x)$ contains a subset of the form $f^{-1}(U) \sqcup U$ for some neighborhood $U \ni x$ by (3), i.e., it contains $p^{-1}(U)$ for some neighborhood U of x. Next, for a point $x \in f(Y)$, let V be a neighborhood of $p^{-1}(x)$. Then since V is a neighborhood of x, by (2) it contains a subset of the form $(f^{-1}(U) - N) \sqcup U$ for some neighborhood U of x and some $N \subset f^{-1}(U)$ with $N \to U$ a closed map. Thus, we find a neighborhood U' of x contained in U such that $f^{-1}(U') \cap N \subset V \cap N$. It follows that $p^{-1}(U') \subset V$. Therefore, we conclude that p is a closed map. ☐

Lemma 3.1.3 (Künneth equivalence)**.** *Let f be a locally universally closed map and consider a cartesian square of topological spaces as follows.*

$$\begin{tikzcd}
Y' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

Then the following square is cocartesian in $\mathsf{Locale}^{\mathrm{L,lex}}$.

$$\begin{tikzcd}
\mathrm{Open}(S) \arrow[r, "g^*"] \arrow[d, "f^*"] & \mathrm{Open}(S') \arrow[d] \\
\mathrm{Open}(Y) \arrow[r] & \mathrm{Open}(Y')
\end{tikzcd}$$

Consequently, we have a cocartesian square in $\mathsf{CAlg}(\mathsf{Pr}^{\mathrm{L}}_{\mathrm{st}})$ *which exhibits an equivalence*

$$\mathsf{Sh}(S' \times_S Y) \simeq \mathsf{Sh}(S') \otimes_{\mathsf{Sh}(S)} \mathsf{Sh}(Y)$$

by [Aok23, Corollary 1.10].

Proof. The proof given here is a slight variant of the argument given in [Lur09, Proposition 7.3.1.11]. In fact, only the final step of our argument differs from the proof of [Lur09, 7.3.1.11].

Let $\mathcal{L}$ denote the pushout $\mathrm{Open}(S') \otimes^{\mathrm{lex}}_{\mathrm{Open}(S)} \mathrm{Open}(Y)$ in $\mathsf{Locale}^{\mathrm{L,lex}}$, which is equipped with the maps $\psi_g\colon \mathrm{Open}(S') \to \mathcal{L}$, $\psi_f\colon \mathrm{Open}(Y) \to \mathcal{L}$ exhibiting the cocartesian square. We let the element $\psi_g(U') \cap \psi_f(V) \in \mathcal{L}$ be denoted by the symbol $U' \otimes_S V$. Since $\psi_g \circ g^*$ and $\psi_f \circ f^*$ commute, we have

$$(U' \cap g^{-1}(U)) \otimes_S V = U' \otimes_S (f^{-1}(U) \cap V).$$

Since it is the pushout in $\mathsf{Locale}^{\mathrm{L,lex}}$, every element of $\mathcal{L}$ is of the form $\bigcup_\alpha U'_\alpha \otimes_S V_\alpha$.

Let $\phi\colon \mathcal{L} \to \mathrm{Open}(S' \times_S Y)$ be the map in $\mathsf{Locale}^{\mathrm{L,lex}}$ provided by the universal property of $\mathcal{L}$. We define a (left-exact but not necessarily colimit-preserving) functor $\theta\colon \mathrm{Open}(S' \times_S Y) \to \mathcal{L}$ by the formula

$$\theta(W) = \bigcup_{U' \times_S V \subset W} U' \otimes_S V.$$

We can easily see that $\phi\theta = \mathrm{id}$. Therefore, it suffices to show that $\theta\phi$ is also the identity. That is, we wish to show that

$$\bigcup_{U' \times_S V \subset \bigcup_\alpha U'_\alpha \times_S V_\alpha} U' \otimes_S V = \bigcup_\alpha U'_\alpha \otimes_S V_\alpha.$$

Here the right hand side is $\leq$ the left hand side by trivial reason. So we claim the converse inequality.

Since $f\colon Y \to S$ is locally universally closed, any open subset V can be written as $\bigcup_\beta T^{\mathrm{int}}_\beta$ where $T_\beta \subset V$ and the map f restricts to a universally closed map $T_\beta \to U_\beta$ for some open subset $U_\beta \subset S$. Since $U' \otimes_S V = \bigcup_\beta U' \otimes_S T^{\mathrm{int}}_\beta$, it suffices to prove that $U' \otimes_S T^{\mathrm{int}}_\beta \subset \bigcup_\alpha U'_\alpha \otimes_S V_\alpha$ whenever $U' \times_S T^{\mathrm{int}}_\beta \subset \bigcup_\alpha U'_\alpha \times_S V_\alpha$.

Fix β. We may replace U' by $U'\cap g^{-1}(U_\beta)$ because $(U'\cap g^{-1}(U_\beta))\otimes_S T_\beta^{\mathrm{int}} = U'\otimes_S (f^{-1}(U_\beta)\cap T_\beta^{\mathrm{int}}) = U'\otimes_S T_\beta^{\mathrm{int}}$. So we may assume that $U'\subset g^{-1}(U_\beta)$.

Fix a point $x'\in U'$ and set $x = g(x')\in U_\beta$. Since $\{x'\}\times_S T_\beta = \{x'\}\times (f^{-1}(x)\cap T_\beta)$ is quasicompact, there exist finitely many $\alpha_1,\dots,\alpha_n$ such that $x'\in\bigcap_{i=1}^n U'_{\alpha_i}$ and

$$f^{-1}(x)\cap T_\beta\subset V_{\alpha_1}\cup\cdots\cup V_{\alpha_n}.$$

Since the map $T_\beta\to U_\beta$ is closed, we can find an open neighborhood U_0 of x in U such that

$$f^{-1}(U_0)\cap T_\beta\subset V_{\alpha_1}\cup\cdots\cup V_{\alpha_n}.$$

Let $U'(x')$ denote the open neighborhood $g^{-1}(U_0)\cap\bigcap_{i=1}^n U'_{\alpha_i}$ of x'. Then we have

$$\begin{aligned} U'(x')\otimes_S T_\beta^{\mathrm{int}} &= \Big(\bigcap U'_{\alpha_i}\Big)\otimes_S \Big(f^{-1}(U_0)\cap T_\beta^{\mathrm{int}}\Big)\\ &\le \Big(\bigcap U'_{\alpha_i}\Big)\otimes_S \bigcup_i V_{\alpha_i}\\ &= \bigcup_i \Big(\bigcap_j U'_{\alpha_j}\Big)\otimes_S V_{\alpha_i}\le \bigcup_i U'_{\alpha_i}\otimes_S V_{\alpha_i}\ \le\ \bigcup_\alpha U'_\alpha\otimes_S V_\alpha. \end{aligned}$$

Taking the colimit in $x'\in U'$, we get $U'\otimes_S T_\beta^{\mathrm{int}}\le\bigcup_\alpha U'_\alpha\otimes_S V_\alpha$ as desired. □

3.2. **Open/closed immersions.**

Theorem 3.2.1 (Open-closed recollement)**.** *Let $j\colon U\hookrightarrow Y$ be an open subspace and $i\colon Z\hookrightarrow Y$ the closed complement. Then:*

1. *The functor $j^*\colon \mathsf{Sh}(Y;\mathsf{An})\to\mathsf{Sh}(U;\mathsf{An})$ admits a left adjoint $j_!$, which is fully faithful and satisfies the projection formula, i.e., the map*

   $$j_!((-)\times_{j^*(-)} j^*(-))\to j_!(-)\times_{(-)}(-)$$

   is an isomorphism.

   In particular, the functor $j^*\colon\mathsf{Sh}(Y)\to\mathsf{Sh}(U)$ admits a fully faithful left adjoint whose oplax $\mathsf{Sh}(Y)$-linear structure is strongly $\mathsf{Sh}(Y)$-linear.
2. *The functor $i_*\colon\mathsf{Sh}(Z;\mathsf{An})\to\mathsf{Sh}(Y;\mathsf{An})$ is fully faithful, preserves filtered colimits. The functor $i^*\colon\mathsf{Sh}(Y;\mathsf{An})\to\mathsf{Sh}(Z;\mathsf{An})$ exhibits $\mathsf{Sh}(Z;\mathsf{An})$ as a localization of $\mathsf{Sh}(Y;\mathsf{An})$ with respect to the maps of the form $\varnothing\to j_!(-)$.*

   In particular, the functor $i_*\colon\mathsf{Sh}(Z)\hookrightarrow\mathsf{Sh}(Y)$, which is defined as the right adjoint to i^*, admits a right adjoint $i^!\colon\mathsf{Sh}(Y)\to\mathsf{Sh}(Z)$.
3. *In $\mathsf{Sh}(Y)$, we have the following fiber sequences.*

   $$j_!j^*\to\mathrm{id}\to i_*i^*$$
   $$i_*i^!\to\mathrm{id}\to j_*j^*$$

   (In fact, each of the fiber sequences can be obtained as the adjoints to the other fiber sequence.) In other words, the null sequence

   $$\mathsf{Sh}(U)\xrightarrow{j_!}\mathsf{Sh}(Y)\xrightarrow{i^*}\mathsf{Sh}(Z)$$

   is a bifiber sequence in $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

Proof. The unstable claims are the combination of [Lur09, §6.3.5, §7.3.2]. The other stable claims are then formal consequences since $\mathsf{Sh}(-)=\mathsf{Sh}(-;\mathsf{An})\otimes\mathsf{Sp}$. □

From now on, we will always consider the categories of sheaves of spectra. In particular, the functors such as $f^*, f_*, j_!, i^!$ will always take values in the categories of sheaves of spectra.

Proposition 3.2.2 (Smooth basechange)**.** *Let*

$$\begin{tikzcd} V \arrow[r, "j'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ U \arrow[r, "j"] & S \end{tikzcd}$$

be a cartesian square of topological spaces in which j is an open embedding. Then the map (in $\mathsf{Sh}(U)$*)*

$$j'_! f'^* \to f^* j_!$$

is an isomorphism.

Proof. This is [Vol23, Lemma 3.25]. One can also prove it using the unstable version [Lur09, Remark 6.3.5.8] together with the observation that both sides are left adjoint functors. ☐

3.3. Proper pushforward.

Theorem 3.3.1 (Proper basechange)**.** *Let*

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "p"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a cartesian diagram of topological spaces in which p is proper. Then the map

$$g^* p_* \to p'_* g'^*$$

is an isomorphism. Moreover, the functor p_* preserves colimits and thus admits a right adjoint $p^!$.

Proof. Combine [MW24] and [Hai22, Proposition 3.8]. We note that although the notion of proper morphisms of ∞-topoi is defined in terms of pullbacks in Top_∞, the pullbacks of 0-localic ∞-topoi are given as the pushouts in $\mathsf{Locale}^{\mathrm{L,lex}}$, so that we have the correct cartesian square of the associated ∞-topoi due to Theorem 3.1.3: ☐

$$\begin{tikzcd} \mathsf{Sh}(Y';\mathsf{An}) \arrow[r, "g'_*"] \arrow[d, "p'_*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Sh}(Y;\mathsf{An}) \arrow[d, "p_*"] \\ \mathsf{Sh}(S';\mathsf{An}) \arrow[r, "g_*"] & \mathsf{Sh}(S;\mathsf{An}) \end{tikzcd} \in \mathsf{Top}_\infty$$

Lemma 3.3.2. *Let*

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "p"] \\ U \arrow[r, "j"] & S \end{tikzcd}$$

be a cartesian square where j is an open embedding and p is proper. Then the map

$$j_! p'_* \to p_* j'_!$$

is an isomorphism

Proof. Let $i\colon Z \to S$ be the closed complement to the map j. Since j^* and i^* are jointly conservative, it suffices to show that the two maps

$$i^* j_! p'_* \to i^* p_* j'_! \qquad\qquad p'_* \to j^* p_* j'_!$$

are isomorphisms. First, using the proper basechange and observing that $j^{-1}(Z) = \varnothing$, we immediately see that the both sides of the first map are zero. Next, under the proper basechange $j^* p_* \simeq p'_* j'^*$, the second map is equivalent to the unit map $p'_* \to p'_* j'^* j'_!$, which is an isomorphism since $j'_!$ is fully faithful. ☐

Lemma 3.3.3 (Proper projection formula)**.** *Let $p\colon Y \to S$ be a proper map. Then the map*

$$p_*(-) \otimes (-) \to p_*(- \otimes p^*(-))$$

is an isomorphism.

Proof. Since both sides commute with colimits and finite limits in each variable, and since $\mathsf{Sh}(S)$ is generated under colimits and finite limits by the objects of the form $j_!(\mathbf{1})$ for $j\colon U \hookrightarrow S$ open, it suffices to show that the map

$$p_*(-) \otimes j_!(\mathbf{1}) \to p_*(- \otimes p^* j_!(\mathbf{1}))$$

is an isomorphism. Consider the cartesian square

$$\begin{tikzcd} p^{-1}U \arrow[r, "j'"] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ U \arrow[r, "j"'] & S \end{tikzcd}.$$

Then under the smooth basechange $p^* j_! \simeq j'_! p'^*$, the smooth projection formula for $j'_!$, the isomorphism $p_* j'_! \simeq j_! p'_*$ in Theorem 3.3.2 and the proper basechange $p'_* j'^* \simeq j^* p_*$, the map in question takes the form

$$p_*(-) \otimes j_!(\mathbf{1}) \to j_! j^* p_*,$$

which is exactly the isomorphism given by the smooth projection formula for j. ☐

3.4. **Shriek functors.**

Theorem 3.4.1. *Let E denote the class of separated locally proper maps. Then the functor*

$$(\mathsf{Sh}(-), (-)^*)\colon \mathsf{Top}^{\mathrm{op}} \to \mathsf{CAlg}(\mathsf{Pr}^{\mathrm{L}}_{\mathrm{st}})$$

extends to a six-functor formalism, i.e., a lax symmetric monoidal functor[10] *of the form*

$$\mathsf{Sh}(-)\colon \mathsf{Corr}(\mathsf{Top})_{E,all} \longrightarrow \mathsf{Pr}^{\mathrm{L}}_{\mathrm{st}}$$

such that for any locally proper map f together with a decomposition $f = pj$ for j an open embedding and p a proper map, the functor $f_!$ is identified with $p_* j_!$.

The right adjoint to a functor of the form $f_!$ will be denoted by $f^!$.

Proof. We provide such a functor by means of [Man22, Proposition A.5.10]. Combining Theorem 3.1.2, Theorem 3.2.1, Theorem 3.2.2, Theorem 3.3.1, Theorem 3.3.2 and Theorem 3.3.3, we see the assumptions of loc. cit. are all satisfied. This completes the proof. ☐

Corollary 3.4.2 ([Man22, Proposition A.5.8])**.** *For a separated locally proper map f, we have the projection formula:*

$$f_! \textit{ is } f^*\textit{-linear,}$$

the basechange theorem

$$g^* f_! \simeq f'_! g'^*,$$

and the functoriality

$$f_{1!} f_{0!} \simeq (f_1 f_0)_!.$$

Here g is any map and f' is the basechange of the map f along g.

3.5. **Homotopy and Monodromy.**

Theorem 3.5.1 ($\mathbb{A}^1$-invariance)**.** *For any topological space S, the map $\mathrm{pr}\colon S \times [0,1] \to S$ induces a fully faithful functor*

$$\mathsf{Sh}(S) \xrightarrow[\mathrm{pr}^*]{} \mathsf{Sh}(S \times [0,1]).$$

Proof. This is a variant of [Lur17, Lemma A.2.9]. Although it proves the unstable version, since the functor $\mathrm{pr}^*\colon \mathsf{Sh}(S \times [0,1]; \mathsf{An}) \to \mathsf{Sh}(S; \mathsf{An})$ admits a left adjoint $\mathrm{pr}_\sharp$, it follows that $\mathrm{pr}^*\colon \mathsf{Sh}(S) \to \mathsf{Sh}(S \times [0,1])$ is fully faithful as well. ☐

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[10] Here we used the notations from [Man22, A.5.2, A.5.4].

Corollary 3.5.2. *Let two map f and g from p to q be homotopic over the base S as in the following diagram.*

$$\begin{tikzcd}
 & X \arrow[dl, "\mathrm{id}"'] \arrow[d, "i_0"] \arrow[dr, "f"] & \\
X \arrow[drr, bend right, "p"'] & X \times [0,1] \arrow[l, "\mathrm{pr}"'] \arrow[r, "h"] & Y \arrow[d, bend left, "q"] \\
 & X \arrow[ul, "\mathrm{id}"] \arrow[u, "i_1"'] \arrow[ur, "g"'] & \\
 & & S
\end{tikzcd}$$

Then we have the following diagram consisting of adjunction units.

$$\begin{tikzcd}
 & p_*\mathrm{pr}_*i_{0*}i_0^*\mathrm{pr}^*p^* = q_*f_*f^*q^* & \\
p_*p^* \arrow[ur, equal] \arrow[dr, equal] \arrow[r, "\sim"] & p_*\mathrm{pr}_*\mathrm{pr}^*p^* \arrow[u] \arrow[d] & q_*q^* \arrow[l] \arrow[ul] \arrow[dl] \\
 & p_*\mathrm{pr}_*i_{1*}i_1^*\mathrm{pr}^*p^* = q_*g_*g^*q^* &
\end{tikzcd}$$

The diagram says that the two maps f^* and g^* on the cohomology (relative to S)

$$f^*, g^* \colon \mathrm{H}^{-\bullet}(Y/S; -) \rightrightarrows \mathrm{H}^{-\bullet}(X/S; -)$$

coincide (witnessed by the above homotopy diagram).

Definition 3.5.3. We say a map $q \colon Y \to S$ is *contractible* if there exist a section $e \colon S \to Y$ and a homotopy $eq \sim \mathrm{id}_Y$ over S.

Note that if q is contractible, then any section s will also serve as e in the above definition.

Corollary 3.5.4. *Let $q \colon Y \to S$ be a contractible map. Then the functor q^* is fully faithful.*

Thus, for a contractible morphism q, we have an isomorphism $q_*q^* \simeq s^*q^*$. In the following, we provide a sufficient condition for a sheaf to be in the image of q^*.

Definition 3.5.5. Let $q \colon Y \to S$. Let us say an object $A \in \mathsf{Sh}(Y)$ is *constant along* q if it belongs to the essential image of q^*. Similarly, A is said to be *locally constant along* q if there exists an open cover $j \colon \coprod U_\alpha \to Y$ such that each $j_\alpha^*(A) \in \mathsf{Sh}(U_\alpha)$ is constant along $qj_\alpha \colon U_\alpha \to S$.

According to the definition, we obtain two full subcategories

$$\mathsf{Sh}_{\mathrm{cnst}/S}(Y) \subset \mathsf{Sh}_{\mathrm{l.c.}/S}(Y) \subset \mathsf{Sh}(Y)$$

spanned by constant and locally constant objects along q, respectively.

Lemma 3.5.6. *Suppose that $q \colon Y \to S$ is a Banach (or more generally locally convex) vector bundle. Let $A \in \mathsf{Sh}(Y)$ be locally constant along q. Then it is constant along q.*

The same holds if q is a composite $Y \hookrightarrow E \to S$ where $E \to S$ is a Banach vector bundle and $Y \subset E$ is locally trivialized as a product of an open in S and a convex subspace of a fiber of $E \to S$. For example, it works for $q = \mathrm{pr} \colon S \times [0,1] \to S$.

Proof. First, we observe that in the following commutative diagram with j an open embedding

$$\begin{tikzcd}
V \arrow[r] \arrow[d, "p"'] \arrow[dr, "q_V"] & Y \arrow[d, "q"] \\
U \arrow[r, hook, "j"'] & S
\end{tikzcd}$$

we have $q_V^* = p^*j^*$ and also $p^* \simeq p^*j^*j_* = q_V^*j_*$ since j_* is fully faithful. Thus, we have an equality $\mathsf{Sh}_{\mathrm{l.c.}/S}(V) = \mathsf{Sh}_{\mathrm{l.c.}/U}(V)$ in this situation. Note that since q is an open map, the above U can be $q(V)$.

Consider the sheaves[11]

$$\mathsf{Sh}(-)\colon \mathrm{Open}(Y)^{\mathrm{op}} \to \widehat{\mathsf{Cat}}$$
$$\mathsf{Sh}(-)\colon \mathrm{Open}(S)^{\mathrm{op}} \to \widehat{\mathsf{Cat}}.$$

The subfunctor

$$\mathsf{Sh}_{\mathrm{l.c.}/S}(-)\colon \mathrm{Open}(Y)^{\mathrm{op}} \to \widehat{\mathsf{Cat}}$$

is also a sheaf by definition. Let $\mathcal{C}(-)\colon \mathrm{Open}(Y)^{\mathrm{op}} \to \widehat{\mathsf{Cat}}$ denote the sheaf obtained by applying the functor $q^*\colon \mathsf{Sh}(S;\widehat{\mathsf{Cat}}) \to \mathsf{Sh}(Y;\widehat{\mathsf{Cat}})$ to the sheaf $\mathsf{Sh}(-)\colon \mathrm{Open}(S)^{\mathrm{op}} \to \widehat{\mathsf{Cat}}$. We have a comparison map $\mathcal{C}(-) \to \mathsf{Sh}_{\mathrm{l.c.}/S}(-)$ by adjunction $q^* \dashv q_*$. For an open subset $V \subset Y$, $\mathcal{C}(V)$ is given by $q_{V*}q_V^*(\mathsf{Sh}(-))(q(V))$, which is equivalent to $\mathsf{Sh}(q(V))$ if $V \to q(V)$ is contractible (by the unstable homotopy invariance).

We claim that the map $\mathcal{C}(-) \to \mathsf{Sh}_{\mathrm{l.c.}/S}(-)$ is an equivalence, which in particular implies that

$$q^*\colon \mathsf{Sh}(S) \simeq \mathcal{C}(Y) \to \mathsf{Sh}_{\mathrm{l.c.}/S}(Y)$$

is an equivalence. By descent, it suffices to show that $\mathcal{C}(q^{-1}U) \to \mathsf{Sh}_{\mathrm{l.c.}/S}(q^{-1}U) = \mathsf{Sh}_{\mathrm{l.c.}/U}(q^{-1}U)$ is an equivalence for $q^{-1}U \cong U \times \mathcal{H}$. Therefore, we may assume that $q\colon Y \to S$ is trivial: $S \times \mathcal{H} \to S$.

Let $M \in \mathsf{Sh}(S \times \mathcal{H})$ be locally constant over S. We take an open cover $\{V_i\}$ of $S \times \mathcal{H}$ of the form $V_i = U_i \times B_i$ for B_i convex open in $\mathcal{H}$ such that M is pulled-back to be constant over $\coprod V_i$. Each of the functor $\mathsf{Sh}(U_{i_0} \cap \cdots \cap U_{i_n}) \to \mathsf{Sh}(V_{i_0} \cap \cdots \cap V_{i_n})$ is fully faithful since $V_{i_0,\dots,i_n} \to U_{i_0,\dots,i_n}$ is contractible. The fully faithful embedding

$$\begin{aligned}\mathcal{C}(S\times\mathcal{H}) &\xrightarrow{\sim} \lim_{[n]\in\mathbf{\Delta}} \prod_{i_0,\dots,i_n} \mathcal{C}(V_{i_0}\cap\cdots\cap V_{i_n})\\ &\xleftarrow{\sim} \lim_{\mathbf{\Delta}} \prod \mathsf{Sh}(U_{i_0}\cap\cdots\cap U_{i_n}) \hookrightarrow \lim_{\mathbf{\Delta}} \prod_{i_0\dots,i_n} \mathsf{Sh}_{\mathrm{l.c.}/S}(V_{i_0}\cap\cdots\cap V_{i_n}) \xleftarrow{\sim} \mathsf{Sh}_{\mathrm{l.c.}/S}(S\times\mathcal{H})\end{aligned}$$

contains M in its image by construction. Therefore, the map $\mathcal{C}(S\times\mathcal{H}) \to \mathsf{Sh}_{\mathrm{l.c.}/S}(S\times\mathcal{H})$ is fully faithful and essentially surjective. □

Corollary 3.5.7. *Let $q\colon Y \to S$ be as in Theorem 3.5.6 and $s\colon S \to Y$ be a section. Then we have an isomorphism*

$$q_* \simeq s^*$$

on $\mathsf{Sh}_{\mathrm{l.c.}/S}(Y)$ the locally constant sheaves relative to S.

Proof. Consider the map $q_* = s^*q^*q_* \to s^*$. It is an isomorphism after precomposed with q^* as we have already seen. This suffices for the proof. □

We record the monodromy equivalence theorem, which tells us that the category of sheaves contains local systems as its full subcategory. We do not use this theorem in the paper, but we would like to regard it as a motivation or justification to use sheaves of spectra.

Theorem 3.5.8 ([Lur17, A.1.15])**.** *Let* $\mathrm{LocSys}\colon \mathsf{Pro}(\mathsf{An}) \to \mathsf{Top}_\infty^{\mathrm{R}}$ *be the right Kan extension of the functor* $(\mathcal{P}(-),(-)_*)\colon \mathsf{An} \to \mathsf{Top}_\infty$*. Since it admits a left adjoint* [Lur09, 7.1.6.15]*, we let* Π_∞ *denote the left adjoint* $\mathsf{Top}_\infty \to \mathsf{Pro}(\mathsf{An})$*.*

Let S be a locally contractible and hypercomplete topological space. Then:

1. $\Pi_\infty S$ *is pro-constant. It is equivalent to the singular simplicial set* $\mathrm{Sing}(S)$ *if, for example, S is homotopy equivalent to a CW-complex*[12]*.*
2. *The functor* $\mathsf{Sh}(S;\mathsf{An}) \to \mathsf{Fun}(\Pi_\infty S,\mathsf{An})$ *induces an equivalence*

$$\mathsf{Sh}_{\mathrm{l.c.}}(S;\mathsf{An}) \xrightarrow{\sim} \mathsf{Fun}(\Pi_\infty S,\mathsf{An})$$

where the left hand side is the full subcategory spanned by the locally constant objects.

---

[11] Follows from *descent* [Lur09, Theorem 6.1.3.9] for ∞-topoi.

[12] More generally, $\Pi_\infty S \simeq \mathrm{Sing}(S)$ if S is paracompact, locally contractible and hypercomplete.

3.6. Duality results.

Notation 3.6.1. We would like to use some ad hoc terminology concerning cohomological smoothness.

1. (1) A map $p\colon Y \to S$ is said to be *topologically smooth* if it is an open map and for every point $y \in Y$ there exists a neighborhood V of y and there exists a homeomorphism $V \cong p(V) \times \mathbb{R}^d$ such that the map $p|_V$ agrees with the composite $V \cong p(V) \times \mathbb{R}^d \xrightarrow{\mathrm{pr}} p(V)$.
2. (2) By a *topological regular immersion*, we mean a closed embedding $i\colon Z \hookrightarrow Y$ such that there exists an open neighborhood N of i such that the map $Z \to N$ is homeomorphic to the zero section $Z \to E$ for some vector bundle E over Z.

Construction 3.6.2. We define Thom spectrum sheaves for (relative tangent) microbundles, which is a variant of Thom spectrum sheaves for vector bundles Theorem 2.1.8.

Let $p\colon Y \to S$ be topologically smooth. Then define $\mathrm{Th}(T_p)$ to be $\pi_{1\sharp}\Delta_!(\mathbf{1})$ where $\Delta\colon Y \to Y \times_S Y$ is the diagonal, $\pi_1\colon Y \times_S Y \to Y$ is the first projection, and $\pi_{1\sharp}$ is left adjoint to π_1^*, which exists by the argument of Theorem 3.6.5.

Remark 3.6.3. To justify our notation, let $p\colon Y \to S$ be a C^1-submersion between, say, C^1-manifolds. Then we have the relative tangent vector bundle $q\colon T_p \to Y$ with zero section $e\colon Y \to T_p$, and there exists a tubular neighborhood of the diagonal Δ, i.e., an open embedding $j\colon T_p \to Y \times_S Y$ such that the diagram

$$\begin{tikzcd}
Y \arrow[r, "\Delta"] \arrow[d, "e"'] & Y \times_S Y \arrow[d, "\pi_1"] \\
T_p \arrow[ur, hook, "j"] \arrow[r, "q"'] & Y
\end{tikzcd}$$

commutes. Therefore, we have an isomorphism $\pi_{1\sharp}\Delta_!(\mathbf{1}) \simeq q_\sharp e_!(\mathbf{1})$ since $j_! = j_\sharp$. Furthermore, the latter $q_\sharp e_!(\mathbf{1})$ is also of the form $q_!(q^!(\mathbf{1}) \otimes e_!(\mathbf{1}))$ by Theorem 3.6.5, the projection formula implies $q_!(q^!(\mathbf{1}) \otimes e_!(\mathbf{1})) \simeq q_! e_! e^* q^!(\mathbf{1}) = e^* q^!(\mathbf{1})$, and $e^* q^!(\mathbf{1}) \simeq q_* q^!(\mathbf{1})$ by Theorem 3.5.7.

Remark 3.6.4. Let Y (and Y') be a locally compact space so that we have the Künneth equivalence $\mathsf{Sh}(S \times Y) \xleftarrow{\sim} \mathsf{Sh}(S) \otimes \mathsf{Sh}(Y)$. By the projection formula, it follows that the functor

$$(f \times g)_!\colon \mathsf{Sh}(S' \times Y') \to \mathsf{Sh}(S \times Y)$$

is identified with $f_! \otimes g_!$ through the Künneth equivalence.

Therefore, whenever we have an isomorphism of the form $g^!(\mathbf{1}) \otimes g^*(-) \simeq g^!(-)$, which in particular implies that $g^!$ admits a further right adjoint, we identify the functor $(\mathrm{id}_S \times g)^!$ with $\mathrm{id}_{\mathsf{Sh}(S)} \otimes g^!$. In other words, $(\mathrm{id}_S \times g)^! \simeq \mathrm{pr}_2^* g^!$ where pr_2 is the projection $S \times Y' \to Y'$.

Proposition 3.6.5 (Poincaré duality)**.** *A topologically smooth morphism p is cohomologically smooth in the following sense: p is $!$-able, $p^!(\mathbf{1})$ is locally constant, $\otimes$-invertible, the map*[13] *$\pi_1^* p^!(\mathbf{1}) \to \pi_2^!(\mathbf{1})$ is an isomorphism and the map*

$$p^!(\mathbf{1}) \otimes p^*(-) \to p^!(-)$$

is an isomorphism. Moreover, $p^!(\mathbf{1})$ is identified with the Thom spectrum sheaf $\mathrm{Th}(T_p)$ of the tangent microbundle (Theorem 3.6.2).

Proposition 3.6.6 (Relative purity)**.** *Let $i\colon Z \to Y$ be a topological regular immersion. Then $i^!(\mathbf{1})$ is locally constant, $\otimes$-invertible and is given as $\mathrm{Th}(-N_i)$ the negative Thom spectrum sheaf of the normal sheaf, i.e., the $\otimes$-inverse of $\mathrm{Th}(N_i)$.*

---

[13] $$\begin{tikzcd}
\cdot \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, "p"] \\
\cdot \arrow[r, "p"'] & \cdot
\end{tikzcd}$$

Proof of Theorem 3.6.5 and Theorem 3.6.6. Both proofs can essentially be found in [Vol23].

In fact, to see that the maps $\pi_1^*p^!(\mathbf{1}) \to \pi_2^!(\mathbf{1})$ and $p^!(\mathbf{1}) \otimes p^*(-) \to p^!(-)$ are isomorphisms, by descent it suffices to show for the trivial vector bundle $\mathbb{R}^n \times S \to S$, which in turn follows by Theorem 3.6.4 from the case for the map $\mathbb{R}^n \to *$. This case is already proved in [Vol23]. Moreover, the isomorphism $p^!(\mathbf{1}) \simeq \mathrm{Th}(T_p)$ can be obtained as follows. By $\pi_1^!(\mathbf{1}) \otimes \pi_1^* \simeq \pi_1^!$, we have $\pi_{1\sharp} = \pi_{1!}(\pi_1^!(\mathbf{1}) \otimes -)$. By the projection formula, $\pi_{1!}(\pi_1^!(\mathbf{1}) \otimes \Delta_!(\mathbf{1}) \simeq \Delta^*\pi_1^!(\mathbf{1})$. Finally, by $\pi_1^!(\mathbf{1}) \simeq \pi_2^*p^!(\mathbf{1})$, we have $\pi_{1\sharp}\Delta_!(\mathbf{1}) \simeq p^!(\mathbf{1})$.

For relative purity, let $i\colon Z \to X$ factor as the zero-section $e\colon Z \to N$ of some vector bundle $p\colon N \to Z$, followed by an open embedding $j\colon N \to X$. Then $i^!(\mathbf{1}) = e^!j^!(\mathbf{1}) = e^!(\mathbf{1})$. By Theorem 3.6.7, we have $e^!(\mathbf{1}) \otimes e^*r^!(\mathbf{1}) \simeq e^!p^!(\mathbf{1}) = \mathbf{1}$. Thus, $e^!(\mathbf{1}) \simeq i^!(\mathbf{1})$ is $\otimes$-invertible with inverse $e^*p^!(\mathbf{1})$, which is isomorphic to $\mathrm{Th}(N)$ (see Theorem 3.6.3). ☐

Lemma 3.6.7. *Let $f\colon Y \to S$ be any map. Then the map*

$$f^!(\mathbf{1}) \otimes f^*(-) \to f^!(-)$$

is an isomorphism on $\mathsf{Pic}(\mathsf{Sh}(S))$ *the $\otimes$-invertible objects.*

Proof. We first note that a $\otimes$-invertible object $E \in \mathsf{Sh}(S)$ is dualizable and the inverse is identified with the dual. Also, the symmetric monoidal functor f^* preserves duality data. In particular, the functors $(-) \otimes E$ and $(-) \otimes f^*E$ admit left adjoints $(-) \otimes E^{-1}$ and $(-) \otimes f^*E^{-1}$, respectively. Passing to the left adjoints of the both sides of the map

$$f^!(-) \otimes f^*(E) \to f^!(- \otimes E)$$

we get the map

$$f_!(- \otimes f^*E^{-1}) \leftarrow f_!(-) \otimes E^{-1},$$

which is the isomorphism exhibiting the projection formula. ☐

Lemma 3.6.8. *Let $p\colon X \to S$ be topologically smooth. Then for any $g\colon S' \to S$, the map $g'^*p^!(\mathbf{1}) \to p'^!(\mathbf{1})$, induced from the cartesian square* $\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ S' \arrow[r, "g"'] & S \end{tikzcd}$, *is an isomorphism.*

Let $e\colon S \to E$ be the zero section to a vector bundle $p\colon E \to S$. For any $g\colon S' \to S$, consider the cartesian rectangle

$$\begin{tikzcd}
S' \arrow[r, "g"] \arrow[d, "e'"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "e"] \\
E' \arrow[r, "g'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
S' \arrow[r, "g"'] & S.
\end{tikzcd}$$

Then the map $g^*e^!(\mathbf{1}) \to e'^!(\mathbf{1})$ is an isomorphism.

Consequently, any map $f\colon X \to S$ that is locally on the source decomposed as a composite of a zero section and a topologically smooth morphism also has the isomorphism $g'^*f^!(\mathbf{1}) \to f'^!(\mathbf{1})$.

Proof. The first claim follows from Theorem 3.6.5. See [Sch23], for example. The second claim follows from the natural duality pairing $e^!(\mathbf{1}) \otimes e^*p^!(\mathbf{1}) \simeq \mathbf{1}$ applied to the symmetric monoidal functor g^*.

We show the final claim. By descent, it suffices to see that the map $j'^*g'^*f^!(\mathbf{1}) \to j'^*f'^!(\mathbf{1})$ is an isomorphism for $j\colon U \hookrightarrow X$ open such that fj decomposes into a zero section $e\colon U \to E$

and a smooth morphism $p\colon E \to S$.

$$\begin{tikzcd}
U' \arrow[r, "g''"] \arrow[d, "j'"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "j"] \arrow[dr, "e"] & \\
X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] & E \arrow[dl, "p"] \\
S' \arrow[r, "g"'] & S &
\end{tikzcd}$$

The map $j'^* g'^* f^!(\mathbf{1}) \to j'^* f'^!(\mathbf{1})$ is thus identified with the map

$$g''^* \quad \underbrace{e^! p^!(\mathbf{1})}_{e^!(\mathbf{1}) \otimes e^* p^!(\mathbf{1})} \quad \to \quad \underbrace{e'^! p'^!(\mathbf{1})}_{e'^!(\mathbf{1}) \otimes e'^* p'^!(\mathbf{1})}$$

and hence an isomorphism since g''^* is symmetric monoidal. □

Appendix A. Genuine equivariance

In this section, we are going to provide a portion (Theorem A.2.6) of genuine equivariant six-functor formalism on topological spaces with Lie group actions, which is sufficient to define the genuine equivariant Bauer–Furuta map (Theorem A.2.7).

A.1. **A quick review of genuine equivariant homotopy theory.** Let G be a Lie group. The category of *genuine (proper equivariant) G-spectra* Sp^G is defined to be the presheaves on the (homotopy-coherent nerve of the) topological category of G-orbits G/K with compact stabilizer subgroups, while the representation spheres S^V are formally inverted. It fits into a larger category of *global spectra* GloSp, where an orbit is replaced by a topological stack BK for all compact Lie groups K.

By the work of [CCL24], these constructions are significantly simplified and it is shown that

$$\mathsf{GloAn} \simeq \mathsf{Sh}_{\mathbb{A}^1}(\mathsf{SepSt}).$$

Here, SepSt is the full subcategory of $\mathsf{Sh}(\mathsf{Mfld}; \mathsf{An}_{\leq 1})$ spanned by those stacks satisfying appropriate representability and separated conditions and the subscript $\mathbb{A}^1$ indicates the homotopy invariant sheaves.

For a G-manifold M with proper action, let $M /\!\!/ G \in \mathsf{SepSt}$ denote the quotient stack. Motivated by the above reformulation of global homotopy theory, and also inspired by the definition of equivariant motivic stable homotopy theory [Hoy17], it is reasonable to consider the following category as a genuine equivariant spectral sheaf theory.

$$\mathsf{SH}(M /\!\!/ G) := \mathsf{Sh}_{\mathbb{A}^1}(\mathrm{Sm}^{\mathrm{sep}}_{M /\!\!/ G})[\mathbb{S}^{-V}]$$

Here, the category $\mathrm{Sm}^{\mathrm{sep}}_{M /\!\!/ G}$ consists of separated stacks over $M /\!\!/ G$ whose structure morphisms onto $M /\!\!/ G$ are *representable submersions*. These categories are studied extensively in [Cno24, Part II].

It is possible to unwind the definition and redefine this category, and adopt it even to non-differentiable base spaces. In the following, we take an abstract approach to these categories. Specifically, we will provide a simple axiom that such a topological variant of the site $\mathrm{Sm}^{\mathrm{sep}}_{X /\!\!/ G}$ should satisfy, and show that these axioms are sufficient to prove the proper basechange theorem. A concrete (six-functor) formalism that satisfies our requirement will be studied in a future work, as well as the detailed behavior of the genuine equivariant Bauer–Furuta map.

A.2. **Genuine equivariant homotopy theory.** Fix a Lie group G. Recall that a G-action on a topological space X is proper if and only if the stacky quotient $X /\!\!/ G$ is separated.

Definition A.2.1. We fix a collection of G-equivariant continuous maps between topological spaces, which is referred to as the collection of *G-smooth maps*, that satisfies the following axiom.

(1) Every G-smooth map is an open map.

(2) If a G-map $p\colon Y \to X$ admits an open cover $\{U_i \to Y\}$ consisting of G-invariant open subspaces such that each map $U_i \to p(U_i)$ is G-smooth, then p is G-smooth.
(3) The collection of G-smooth maps is closed under basechange and composition, and it contains all G-equivariant homeomorphisms.
(4) Every (finite-rank) G-equivariant vector bundle is G-smooth.
(5) If K is a compact subgroup of G, H is a closed subgroup of G with $K \subset H$, and S is a topological space with H-action, then the map $\mathrm{ind}_K^G(S) \to \mathrm{ind}_H^G(S)$ is G-smooth.
(6) For each X, the collection of G-smooth maps with target X forms a small category.

Define the site Sm_X^G as follows. The objects are the G-smooth maps with target X, and the morphisms are G-maps over X. The category Sm_X^G is equipped with the Grothendieck topology generated by open coverings consisting of G-invariant open subspaces.

For example, the category $\mathrm{Sm}^G := \mathrm{Sm}^G_{\mathrm{pt}}$, where X is taken to be a point, contains every differentiable G-manifold with proper action, such as an orbit G/K with compact stabilizer subgroup K. This coincides with the convention from *proper equivariant* homotopy theory.

An easy example of a class of G-smooth maps is the class of G-maps whose underlying continuous maps are topological submersions, which could be one of the worst choice due to the absence of local models of G-smooth maps. Thus, a preferred choice for the definition of G-smooth maps would be the smallest collection satisfying the axiom given above. However, the precise definition will not be needed in the following arguments.

Remark A.2.2 (Smooth/open sharp)**.** Let $p\colon Y \to X$ be G-smooth and let $p_\circ$ denote the functor $\mathrm{Sm}_Y^G \to \mathrm{Sm}_X^G$ obtained by post-composing with p. Then we have an adjunction $\mathrm{Sm}_Y^G \underset{p^*}{\overset{p_\circ}{\rightleftarrows}} \mathrm{Sm}_X^G$, each of which is a morphism of sites, meaning that the precomposition functors preserve sheaves. Thus, we obtain a triple as follows

$$\begin{tikzcd}
\mathsf{Sh}(\mathrm{Sm}_Y^G) \arrow[r, shift left=4, "p_\sharp"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=4, "p_*"'] & \mathsf{Sh}(\mathrm{Sm}_X^G) \arrow[l, "p^*" description]
\end{tikzcd}$$

where $p_\sharp$ is the sheafified left Kan extension functor along $p_\circ$. Next, consider the site Open_Y^G of G-invariant open subsets and the functor $\jmath\colon \mathrm{Open}_Y^G \hookrightarrow \mathrm{Sm}_Y^G$, which is a morphism of sites having the covering lifting property[14]. By this, we obtain the following triple

$$\begin{tikzcd}
\mathsf{Sh}(\mathrm{Open}_Y^G) \arrow[r, shift left=4, "\jmath_\sharp"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=4, "\jmath_*"'] & \mathsf{Sh}(\mathrm{Sm}_Y^G) \arrow[l, "\jmath^*" description]
\end{tikzcd}$$

where $\jmath_\sharp$ is the sheafified left Kan extension along $\jmath$ and $\jmath^*$ is the restriction along $\jmath$.

We would like to abuse notation so that $p_\sharp\colon \mathsf{Sh}(\mathrm{Open}_Y^G) \to \mathsf{Sh}(\mathrm{Sm}_X^G)$ also denotes the composite $p_\sharp \jmath_\sharp$.

Lemma A.2.3 (Smooth basechange)**.** *Let $f\colon X' \to X$ be a G-map and $p\colon Y \to X$ be G-smooth. Let Y' denote the pullback $Y \times_X X'$ and let f' and p' be the resulting basechanges. Then the following diagram commutes.*

$$\begin{tikzcd}
\mathsf{Sh}(\mathrm{Open}_{Y'}^G) \arrow[r, "p'_\sharp"] & \mathsf{Sh}(\mathrm{Sm}_{X'}^G) \\
\mathsf{Sh}(\mathrm{Open}_Y^G) \arrow[u, "f'^*"] \arrow[r, "p_\sharp"'] \arrow[ur, phantom, "\Searrow"] & \mathsf{Sh}(\mathrm{Sm}_X^G) \arrow[u, "f^*"']
\end{tikzcd}$$

Proof. Since every functor in the diagram commutes with colimits, it suffices to show that the map

$$p'_\sharp f'^* \jmath_!(\mathbf{1}) \to f^* p_\sharp \jmath_!(\mathbf{1})$$

---

[14] See [Pst23, Appendix A.1] for definition and consequence.

is an isomorphism for any G-invariant open embedding $j\colon U\to Y$. This follows from the following easy commutative diagram.

$$\begin{tikzcd}
\mathrm{Open}^G_{Y'} \arrow[r, "p'(-)"] & \mathrm{Sm}^G_{X'} \\
\mathrm{Open}^G_{Y} \arrow[u, "f'^{-1}"] \arrow[r, "p(-)"'] & \mathrm{Sm}^G_{X} \arrow[u, "f^{-1}"']
\end{tikzcd}$$

□

Proposition A.2.4 (Proper pushforward)**.** *Let $f\colon X'\to X$ be a proper G-map. We assume that G-action on X' is a proper action. Then the functor $f_*\colon \mathsf{Sh}(\mathrm{Sm}^G_{X'})\to\mathsf{Sh}(\mathrm{Sm}^G_X)$ preserves colimits.*

Moreover, the proper projection formula holds, i.e., the map $f_*(-)\otimes(-)\to f_*(-\otimes f^*(-))$ is an isomorphism.

Proof. Since the functors $\left\{p^*\colon \mathsf{Sh}(\mathrm{Sm}^G_X)\to\mathsf{Sh}(\mathrm{Open}^G_Y)\right\}_{p\in\mathrm{Sm}^G_X}$ are jointly conservative and each p^* preserves colimits, it suffices to show that each p^*f_* is colimit-preserving. Fix such a G-smooth map p and the following cartesian square.

$$\begin{tikzcd}
Y' \arrow[r, "p'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[d, "f"] \\
Y \arrow[r, "p"'] & X
\end{tikzcd}$$

By Theorem A.2.3, p^*f_* is equivalent to the following composite functor.

$$\mathsf{Sh}(\mathrm{Sm}^G_{X'})\xrightarrow{p'^*}\mathsf{Sh}(\mathrm{Open}^G_{Y'})\xrightarrow{f'_*}\mathsf{Sh}(\mathrm{Open}^G_{Y})$$

Since p'^* preserves colimits, it suffices to show that f'_* preserves colimits. The functor f'_* is the same as the functor $(f'/G)_*\colon \mathsf{Sh}(Y'/G)\to\mathsf{Sh}(Y/G)$ on the quotient topological spaces, and the map f'/G is separated universally closed by assumption. Therefore, it preserves colimits.

The projection formula can be proved in the same way, by observing that the functors p^* and p'^* preserve tensor products. □

Definition A.2.5. Let X be a topological space with G-action. Define the full subcategory $\mathsf{Sh}_{\mathbb{A}^1}(\mathrm{Sm}^G_X)$ of $\mathsf{Sh}(\mathrm{Sm}^G_X)$ to be the localization of $\mathsf{Sh}(\mathrm{Sm}^G_X)\in\mathsf{CAlg}(\mathsf{Pr}^{\mathrm{L}}_{\mathrm{st}})$ with respect to the $\otimes$-ideal generated by the map $X\times\mathbb{R}^1\to X$ with trivial G-action on $\mathbb{R}^1$.

For each (finite-dimensional) G-representation V, consider the map $p\colon V\to *$ and the zero-section $e\colon *\to V$. Then we have $p_\sharp s_*(\mathbf{1})\in\mathsf{Sh}(\mathrm{Sm}^G)$, which is thought of as the Thom spectrum of the form $\mathbb{S}^V$.

Definition A.2.6. Define the *genuine equivariant homotopy category* $\mathsf{SH}^G_{\mathrm{top}}$ to be the formal inversion

$$\mathsf{Sh}_{\mathbb{A}^1}(\mathrm{Sm}^G)\left[p_\sharp s_*(\mathbf{1})^{-1}\mid V\in\mathsf{Rep}(G)\right]\in\mathsf{CAlg}(\mathsf{Pr}^{\mathrm{L}}).$$

For a topological G-space X, define its genuine equivariant stable homotopy category $\mathsf{SH}^G_{\mathrm{top}}(X)$ to be $\mathsf{Sh}_{\mathbb{A}^1}(\mathrm{Sm}^G_X)\otimes_{\mathsf{Sh}_{\mathbb{A}^1}(\mathrm{Sm}^G)}\mathsf{SH}^G_{\mathrm{top}}$.

For a G-map $f\colon X'\to X$, let $f^*\colon\mathsf{SH}^G_{\mathrm{top}}(X)\to\mathsf{SH}^G_{\mathrm{top}}(X')$ denote the functor obtained by tensoring the functor $f^*\colon\mathsf{Sh}_{\mathbb{A}^1}(\mathrm{Sm}^G_{X'})\to\mathsf{Sh}_{\mathbb{A}^1}(\mathrm{Sm}^G_X)$ with $\mathsf{SH}^G_{\mathrm{top}}$.

For a proper G-map $f\colon X'\to X$, by Theorem A.2.4, the functor f^* is an internal left adjoint functor in $\mathsf{Mod}_{\mathsf{Sh}_{\mathbb{A}^1}(\mathrm{Sm}^G)}(\mathsf{Pr}^{\mathrm{L}})$. Therefore, for such a proper G-map, f^* on $\mathsf{SH}^G_{\mathrm{top}}(-)$ is an internal left adjoint functor, whose right adjoint f_* has a further right adjoint which is denoted by $f^!$.

The formal inversion is essential to ensure that the upper shriek functors compute correct Thom spectra. We finally have the genuine equivariant analogue of Theorem 2.1.1.

Construction A.2.7. Let L, X and S be topological spaces acted on by a Lie group G where we assume L to have proper G-action. Let $f\colon L \to X$ be a proper G-map over S.

$$\begin{tikzcd}
L \arrow[rr, "f"] \arrow[dr, "r"'] & & X \arrow[dl, "q"] \\
& S &
\end{tikzcd}$$

Then we have a map in $\mathsf{SH}^G_{\mathrm{top}}(S)$ of the following form.

$$\mathrm{BF}_f\colon r_*f^!(\mathbf{1}) = q_*f_*f^!(\mathbf{1}) \xrightarrow{q_*(\mathrm{counit})} q_*(\mathbf{1})$$

This is our definition of the *genuine equivariant Bauer–Furuta map.*

Appendix B. The Seiberg–Witten map

B.1. **Recollection of the Seiberg–Witten map.** Let us recall the construction of the Seiberg–Witten map (1.3), following [Fur01] and [Fur97].

In this section, Mfld denotes the category of C^∞-manifolds and C^∞-maps between them. We equip Mfld with the Grothendieck topology generated by open covers. The category $\mathsf{St} := \mathsf{Sh}(\mathsf{Mfld}; \mathsf{An}_{\leq 1})$ of (1-truncated) stacks will be useful. For example, we have a fully faithful embedding $B\colon \mathsf{LieGrp} \subset \mathsf{Grp}(\mathsf{St}) \hookrightarrow \mathsf{St}_*$ given by the delooping functor. All Lie groups, vector spaces and manifolds will be viewed as objects in St. In the slice category $\mathsf{St}_{/\mathcal{Y}}$ over a stack $\mathcal{Y}$, we have $\underline{\mathrm{Hom}}_{\mathcal{Y}}(-,-) \in \mathsf{St}_{/\mathcal{Y}}$ the internal hom and $\underline{\mathrm{Hom}}_{/\mathcal{Y}}(-,-) \in \mathsf{St}$ the St-enrichment.

Consider the Lie group $\mathrm{Spin}^c(n)$, which can be defined as $(\mathrm{Spin}(n) \times U(1))/\mathrm{diag}\{\pm 1\}$. In dimension 4, we have an isomorphism $\mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ which is exhibited by the representation $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4)$ given as follows.

$$\mathrm{Sp}(1) \times \mathrm{Sp}(1) \ni (q_-, q_+) \quad \mapsto \quad q_-(-)q_+^{-1} \curvearrowright \mathbb{H}\ (\cong \mathbb{R}^4)$$

The group $\mathrm{Spin}^c(4)$ acts in the same way (where the $U(1)$-part acts trivially) and the resulting representation is denoted by ${}_-\mathbb{H}_+$. We further need the following three representations of $\mathrm{Spin}^c(4)$.

$$\begin{array}{rcrl}
{}_+\mathbb{H} & : & \mathrm{Sp}(1) \times \mathrm{Sp}(1) \times U(1) \ni & (q_-, q_+, u) \mapsto q_+(-)z^{-1} \curvearrowright \mathbb{H} \\
{}_-\mathbb{H} & : & & (q_-, q_+, u) \mapsto q_-(-)z^{-1} \curvearrowright \mathbb{H} \\
{}_+\mathbb{H}_+ & : & & (q_-, q_+, u) \mapsto q_+(-)q_+^{-1} \curvearrowright \mathbb{H}
\end{array}$$

The two ${}_\pm\mathbb{H}$ are complex representations, where we let the complex number $\sqrt{-1}$ act on ${}_\pm\mathbb{H}$ as the right multiplication by $i \in \mathbb{H}$. Using the orthonormal basis $(1, i, j, ji)$ of $\mathbb{H}$ as an orientation for ${}_-\mathbb{H}_+$, the imaginary part $\Im({}_+\mathbb{H}_+)$, which is invariant under $\mathrm{Spin}^c(4)$, is isomorphic to $\Lambda^{2,+}({}_-\mathbb{H}_+)$ the $(+1)$-eigenspace of the Hodge star operator acting on Λ^2. The projection map from $\Lambda^2({}_-\mathbb{H}_+)$ onto $\Lambda^{2,+}({}_-\mathbb{H}_+) \cong \Im({}_+\mathbb{H}_+)$ will be denoted by the superscript $(-)^+$. We have two (complex and real, respectively) Clifford actions as follows.

$$\begin{array}{rcr}
({}_-\mathbb{H}_+ \otimes_{\mathbb{R}} \mathbb{C}) \otimes {}_+\mathbb{H} \to {}_-\mathbb{H} & ; & (a + b\sqrt{-1}, \phi) \mapsto a\phi + b\phi i \\
{}_-\mathbb{H}_+ \otimes_{\mathbb{R}} {}_-\mathbb{H}_+ \to {}_+\mathbb{H}_+ & ; & (a, b) \mapsto \overline{a}b
\end{array} \tag{B.1}$$

And finally a quadratic form:

$$\sigma\colon {}_+\mathbb{H} \ni \phi \quad \mapsto \quad -\phi i \overline{\phi} \otimes \sqrt{-1} \ \in \Lambda^{2,+}({}_+\mathbb{H}_+) \otimes_{\mathbb{R}} \sqrt{-1}\mathbb{R}$$

Let X be a closed oriented 4-manifold. Take a riemannian metric g and a spin^c structure $\mathfrak{s}$ on X, i.e., $\mathfrak{s}$ is a lift as in the following diagram in St.

$$\begin{tikzcd}
& B\mathrm{Spin}^c(4) \arrow[d] \\
& B\mathrm{SO}(4) \arrow[d] \\
X \arrow[uur, dashed, bend left, "\mathfrak{s}"] \arrow[ur, "g"] \arrow[r, "T^*_X"'] & B\mathrm{GL}^+(4)
\end{tikzcd}$$

In other words, $\mathfrak{s}$ is a pair consisting of a $\mathrm{Spin}^c(4)$-torsor P on X together with an isomorphism of sheaves $\Omega^1_X \cong \underline{\mathrm{Hom}}^{\mathrm{Spin}^c(4)}(P, {}_-\mathbb{H}_+)$ where the right hand side is the mapping stack of $\mathrm{Spin}^c(4)$-equivariant maps. Next, we take a spin^c-connection on the torsor P such that it lifts the riemannian connection on T^*_X associated with the metric g. Such a connection corresponds exactly to a $U(1)$-connection A (on the determinant line bundle of P), since the Lie algebra of $\mathrm{Spin}^c(4)$ splits as a sum of $\mathfrak{so}(4)$ and $\mathfrak{u}(1) = \sqrt{-1}\mathbb{R}$. We abuse notation and let the corresponding $\mathrm{Spin}^c(4)$-connection be denoted also by A.

We construct the *Seiberg–Witten map* as follows. The connection $A + a\sqrt{-1}$ defines the covariant derivative on the spaces such as $\underline{\mathrm{Hom}}^{\mathrm{Spin}^c(4)}(P, {}_{(\pm)}\mathbb{H}_{(+)})$ and thus we can post-compose with the Clifford actions (B.1) to obtain the Dirac-type operator of the following form.

$$\underline{\mathrm{Hom}}^{\mathrm{Spin}^c(4)}\left(P,\ {}_-\mathbb{H}_+ \oplus {}_+\mathbb{H}\right) \to \underline{\mathrm{Hom}}^{\mathrm{Spin}^c(4)}\left(P,\ {}_-\mathbb{H} \oplus ({}_+\mathbb{H}_+ \otimes \sqrt{-1})\right)$$

In fact, it is of the form $(a, \phi) \mapsto \left(D_A\phi + a\phi i,\ d^*a\sqrt{-1} + (da)^+\sqrt{-1}\right)$. Consider another map on the same spaces given by the formula

$$(a, \phi) \mapsto (0, F^+_A - \sigma(\phi))$$

where F_A is the curvature 2-form (valued in $\sqrt{-1}\mathbb{R}$). We also would like to add the locally constant functions on the left hand side, which is mapped to $\underline{\mathrm{Hom}}^{\mathrm{Spin}^c(4)}(P, {}_+\mathbb{H}_+)$ via the inclusion $\mathbb{R} \cong \Re({}_+\mathbb{H}_+) \subset {}_+\mathbb{H}_+$. Combining all together, we obtain the Seiberg–Witten map.

$$\begin{aligned} \mu_{g,\mathfrak{s},A}\colon \underline{\mathrm{Hom}}^{\mathrm{Spin}^c}(P, {}_-\mathbb{H}_+ \oplus {}_+\mathbb{H}) \times H^0_{\mathrm{dR}}(X, \mathbb{R}) &\to \underline{\mathrm{Hom}}^{\mathrm{Spin}^c(4)}(P, {}_-\mathbb{H} \oplus ({}_+\mathbb{H}_+ \otimes \sqrt{-1})) \\ (a, \phi, r) &\mapsto (D_{A+a\sqrt{-1}}\phi,\ r + d^*a + F^+_{A+a\sqrt{-1}} + \phi i \bar{\phi}\sqrt{-1}) \end{aligned} \tag{B.2}$$

For $k > 4$, the map on the Sobolev completions $L^2_k \to L^2_{k-1}$ induced by the linear part, together with the map on L^2_k's induced by the nonlinear part post-composed by the Sobolev embedding $L^2_k \to L^2_{k-1}$, is the map we can apply our Bauer–Furuta construction: See [BF04, Lemma 3.1] and Theorem B.2.2.

Note that the Seiberg–Witten map is parameterized over the space of $U(1)$-connections and has the gauge group equivariance, which we summarize in the following language.

Remark B.1.1 (Gauge group)**.** Consider the stack $B_\nabla\mathrm{Spin}^c(n)$ classifying $\mathrm{Spin}^c(n)$-torsors and connections on them, which is just the quotient stack of the lie algebra $\mathfrak{g}$ acted on by $g \in G$ via the formula $g^{-1}dg + g^{-1}(-)g$. Let $\mathrm{Spin}^c_\nabla(X)$ denote the stack $\underline{\mathrm{Hom}}_{/B\mathrm{GL}^+(4)}(X, B_\nabla\mathrm{Spin}^c(4))$, which roughly classifies a triple $(g, \mathfrak{s}, A)$ consisting of a metric, a spin^c-structure and a spin^c-connection. We similarly consider the stacks $\mathrm{Met}(X) = \underline{\mathrm{Hom}}_{/B\mathrm{GL}^+(4)}(X, B\mathrm{SO}(4))$ and $\mathrm{Met}_\nabla(X) = \underline{\mathrm{Hom}}_{/B\mathrm{GL}^+(4)}(X, B_\nabla\mathrm{SO}(4))$, where the forgetful map $\mathrm{Met}_\nabla(X) \to \mathrm{Met}(X)$ admits a section $\mathrm{LC}\colon \mathrm{Met}(X) \to \mathrm{Met}_\nabla(X)$ given by the riemannian connection. Form the following pullback and obtain our base stack $\mathcal{S}_X$ for the Seiberg–Witten map.

$$\begin{tikzcd} \mathcal{S}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spin}^c_\nabla(X) \arrow[d] \\ \mathrm{Met}(X) \arrow[r, "\mathrm{LC}"'] & \mathrm{Met}_\nabla(X) \end{tikzcd}$$

Fixing a (metric and a) spin^c-structure giving a torsor P, the fiber of the forgetful map $\mathcal{S}_X \to \mathrm{Spin}^c_\nabla(X) \to \mathrm{Spin}^c(X)$ is isomorphic to the moduli stack $\mathcal{A}(\det P)$ of $U(1)$-connections on the determinant line bundle. The based looping $\Omega\mathcal{A}(\det P)$ is called the *gauge group* $\mathcal{G}$, which is isomorphic to the group $\underline{\mathrm{Hom}}(X, U(1))$.

Over the product $X \times \mathcal{S}_X$, we have the $\mathrm{Spin}^c(4)$-torsor, say $\mathcal{P}$, which is pulled-back along the evaluation map $X \times \underline{\mathrm{Hom}}(X, B\mathrm{Spin}^c(4)) \to B\mathrm{Spin}^c(4)$. The Seiberg–Witten map μ can be considered as the map

$$\mu\colon \underline{\mathrm{Hom}}^{\mathrm{Spin}^c(4)}_{\mathcal{S}_X}(\mathcal{P},\ {}_-\mathbb{H}_+ \oplus {}_+\mathbb{H}) \times H^0_{\mathrm{dR}}(X, \mathbb{R}) \to \underline{\mathrm{Hom}}^{\mathrm{Spin}^c(4)}_{\mathcal{S}_X}(\mathcal{P},\ {}_-\mathbb{H} \oplus {}_+\mathbb{H}_+ \cdot \sqrt{-1}) \tag{B.3}$$

over $\mathcal{S}_X$, which is defined by the same formula we presented above. Since the stack $\mathcal{S}_X$ is highly structured, fixing a single spin^c-structure, the map μ restricts to a $\mathcal{G}$-equivariant map, for trivial reason.

Note that we have the Galois action $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \mathrm{Aut}_{\mathsf{Grp}}(\mathrm{Spin}^c(n))$ given by the complex conjugation on the $U(1)$-part of $\mathrm{Spin}^c(n)$. It induces a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action on the stack $\mathrm{Spin}^c_\nabla(X)$. Let $(\mathfrak{s}', A')$ denote the conjugate of $(\mathfrak{s}, A) \in \mathrm{Spin}^c_\nabla(X)$ by this $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action. This gives us a $\mathrm{Pin}(2)$-symmetry of the Seiberg–Witten map (B.3) as follows.

Remark B.1.2 (Pin(2)-symmetry [Fur97])**.** Define order 4 automorphisms J as follows. On the source of μ, J acts by the formula $(\mathfrak{s}, A; a, \phi, r) \mapsto (\mathfrak{s}', A'; -a, \phi j, -r)$, and on the target of μ by the formula $(\mathfrak{s}, A; \varphi, b) \mapsto (\mathfrak{s}', A'; \varphi j, -b)$. The map μ commutes with these J-actions, and J and $u \in H^0_{\mathrm{dR}}(X, U(1))$ satisfy the relations $J \circ u = \overline{u} \circ J$ and $J^2 = -1 \in U(1)$. Therefore, in this way the Seiberg–Witten map μ promotes to a map over the stack $BH^0_{\mathrm{dR}}(X, \mathrm{Pin}(2))$.

If we consider a spin^c-structure $\mathfrak{s}$ coming from a spin structure, then the determinant line bundle is trivial and admits a trivial connection A_0 which is fixed by the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action. Thus, the single Seiberg–Witten map $\mu_{g,\mathfrak{s},A_0}$ (B.2) promotes to a $\mathrm{Pin}(2)$-equivariant map.

Another way of stating this Pin(2)-symmetry is to observe that the representations and maps between them we have used so far are promoted to be equivariant under the larger group $\mathrm{Spin}^{c-}(4) = (\mathrm{Spin}(4) \times \mathrm{Pin}(2))/\mathrm{diag}\{\pm 1\}$ and to replace the base stack accordingly. This perspective was introduced by [Nak13].

The stacky constructions presented above would be readily applicable to our reformulation of the Bauer–Furuta construction *if only* there exists a genuine six-functor formalism, defined over those geometric stacks, which should send BG of a Lie group to the genuine G-spectra and subsume the Atiyah duality for G-manifolds.

B.2. **Some example of properness.** We here deal with an abstract lemma that enables us to conclude the properness of the families Seiberg–Witten map. Essentially, it is a recollection of the arguments given in [BF04]. We continue to assume that all topological spaces are Hausdorff. First, observe the following.

Lemma B.2.1. *Let S be compactly generated and $\pi\colon E \to S$ be a Banach vector bundle. Assume that either of the following holds.*

1. *S is first countable,*
2. *S is locally compact, or*
3. *the local triviality of the fiber bundle π is interpreted in the cartesian closed category of compactly generated topological spaces.*

Then the total space E is compactly generated.

Proof. Since being compactly generated is a local property, we may assume that the total space E is of the form $U \times \mathcal{H}$ for some Banach space $\mathcal{H}$. Note that a finite product of first countable topological spaces (such as U and $\mathcal{H}$) is first countable and hence compactly generated. Also, a product of a compactly generated space $\mathcal{H}$ and a locally compact space U is also compactly generated. ☐

Of course, in the previous lemma, S can be a product of a first countable space and a locally compact space. We note that the homotopy invariance (Theorem 3.5.4 or Theorem 3.5.6) for (Banach) vector bundles remains valid even when we replace the product topology by compactly generated ones, though in certain statements such as Theorem 3.1.3, the choice of product topology seems to be crucial.

We next record a variant of [BF04, Lemma 2.3]. The proof goes basically the same as that of [BF04, Lemma 2.3]. This will serve as a source for the families Bauer–Furuta invariant based on our formalism.

Lemma B.2.2. *Let $f\colon E' \to E$ be a map between Banach vector bundles over a base S which is of the form $f = l + c$ for l a linear Fredholm map and c a (possibly nonlinear) compact map in the sense that it maps a disk bundle over a compact subset $K \subset S$ to a relatively compact* *subset of* $K \times_S E$*. Assume that on each fiber the map* $f_x \colon E'_x \to E_x$ *has bounded preimages of bounded subsets and that the vector bundle* $q \colon E \to S$ *satisfies either of the assumptions in Theorem B.2.1. Then* f *is a proper map.*

Proof. Since properness is local on the target, we may assume that the vector bundles are trivial. Recall that a map to a compactly generated space is proper if and only if the preimages of compact subspaces of the target are (quasi)compact. Therefore, we may assume that the map f is of the form $K \times \mathcal{H}' \to K \times \mathcal{H}$ for K compact. Since the preimage $f^{-1}(y)$ of a point is contained in a bounded neighborhood (in a fiber) by assumption, it suffices to show that for any bounded subset $A \subset \mathcal{H}'$ the restriction of f to $K \times A$ is proper.

Take any point $x_0 \in K$. Choose splittings $\mathcal{H}' \cong \ker(l_{x_0}) \times \operatorname{im}(l_{x_0})$ and $\mathcal{H} \cong \operatorname{im}(l_{x_0}) \times \operatorname{cok}(l_{x_0})$. Replace K by a smaller compact neighborhood of x_0 so that the composite $\mathcal{H}' \xrightarrow{l_x} \mathcal{H} \xrightarrow{\mathrm{pr}} \operatorname{im}(l_{x_0})$ is surjective for every point $x \in K$ (Theorem C.1.5). So we take the kernels of that composite for $x \in K$ and form a (finite-dimensional) vector bundle F over K (Theorem C.1.3). We may assume that the subbundle $F \subset K \times \mathcal{H}'$ splits $\rho \colon K \times \mathcal{H}' \to F$ (since it is finite-dimensional). Both c and ρ are compact maps. The map $f\,|_{K \times A}$ decomposes into the following.

$$K \times A \longrightarrow \mathcal{H} \times \overline{c(K \times A)} \times_K \overline{\rho(K \times A)} \times_K \overline{\rho l(K \times A)} \overset{\cong}{\to} (K \times \mathcal{H}) \times_K \overline{c(K \times A)} \times_K \overline{\rho(K \times A)} \times_K \overline{\rho l(K \times A)} \overset{\mathrm{pr}}{\to} K \times \mathcal{H}$$

$$(x,a) \mapsto \begin{pmatrix} \mathrm{pr}_{\operatorname{im}(l_{x_0})}(l_x(a)) \\ c_x(a) \\ \rho_x(a) \\ \rho_x l_x(a) \end{pmatrix}, \qquad \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \mapsto \begin{pmatrix} x+z+w \\ y \\ z \\ w \end{pmatrix}$$

The first map is a closed embedding since it admits a retraction: $a = \mathrm{pr}_{\operatorname{im}(l_{x_0})}(l_x(a)) + \rho_x(a)$. The third map is proper since $c, \rho, \rho l$ are all compact maps. ☐

Appendix C. Recollection: Banach manifold theory

C.1. **Vector bundles.** A *Banach space* is a complete normed topological vector space (over $\mathbb{R}$). Let $\mathsf{Ban}_{\mathbb{R}}$ denote the category of Banach spaces and continuous linear maps between them.

Convention C.1.1 (Banach/Hilbert bundles)**.** For Banach spaces $\mathcal{H}$ and $\mathcal{H}'$, we put the norm topology on the space of continuous/bounded linear maps $\mathrm{Hom}_{\mathsf{Ban}_{\mathbb{R}}}(\mathcal{H}, \mathcal{H}')$. A pleasant property for this topology is that the evaluation and the composition maps

$$\mathcal{H} \times \mathrm{Hom}_{\mathsf{Ban}_{\mathbb{R}}}(\mathcal{H}, \mathcal{H}') \to \mathcal{H}'$$
$$\mathrm{Hom}_{\mathsf{Ban}_{\mathbb{R}}}(\mathcal{H}, \mathcal{H}') \times \mathrm{Hom}_{\mathsf{Ban}_{\mathbb{R}}}(\mathcal{H}', \mathcal{H}'') \to \mathrm{Hom}_{\mathsf{Ban}_{\mathbb{R}}}(\mathcal{H}, \mathcal{H}'')$$

are continuous. The notion of *Banach vector bundles* (over a topological space) will always be interpreted in this topology: such notion behaves well since there exists a topological group structure on the space of continuous linear automorphisms (Theorem C.1.2). We would like to define a *Hilbert vector bundle* to be a Banach vector bundle whose fibers admit a further structure of a Hilbert space.

We say a (continuous) map between Banach vector bundles a *linear map* if it can be locally trivialized to a continuous family of bounded linear maps (in terms of the norm topology). In the same way, we can consider *linear Fredholm maps* between Banach vector bundles.

The following well-known properties are fundamental.

Proposition C.1.2. *Let* $\mathcal{H}$ *and* $\mathcal{H}'$ *be Banach spaces. Then:*

(1) *A continuous linear bijection* $\mathcal{H} \to \mathcal{H}'$ *admits a continuous inverse, i.e., it is an isomorphism in the category* $\mathsf{Ban}_{\mathbb{R}}$*.*
(2) *The subspace* $\mathrm{Iso}_{\mathsf{Ban}_{\mathbb{R}}}(\mathcal{H}, \mathcal{H}')$ *of continuous linear isomorphisms is open in the norm topology.*
(3) *The group* $\mathrm{Aut}_{\mathsf{Ban}_{\mathbb{R}}}(\mathcal{H})$ *of continuous linear automorphisms is a topological group in the norm topology, in the sense that the inverse operation is also continuous.*

The following properties are immediate consequences.

Corollary C.1.3. *Let $E \to E'$ be a linear map of Hilbert vector bundles over a base S. If it is surjective on fibers, then the kernels form a Hilbert vector bundle.*

Even if we consider a linear map $\varphi\colon E \to E'$ of Banach vector bundles, the kernels form a Banach vector bundle whenever the map is surjective on fibers and the (fiberwise) kernels $\ker(\varphi_x)$ *have complementary closed linear subspaces in those fibers E_x.*

Corollary C.1.4. *Let $E \to E'$ be a linear map of Banach vector bundles over a base S. If it is injective on fibers and the (fiberwise) images* $\mathrm{im}(\varphi_x)$ *have complementary closed linear subspaces in those fibers E_x, then the cokernels form a Banach vector bundle.*

Corollary C.1.5. *Let $\varphi\colon E \to E'$ be a linear map between Banach vector bundles over S. Assume that each kernel* $\ker(\varphi_x)$ *admits a complementary closed linear subspace in E_x. Then the subset $\{x \in S \mid \varphi_x$ is a surjection$\}$ is open in S.*

C.2. **Some calculus on vector bundles.** We provide a variant of the inverse function theorem for Banach vector bundles.

Definition C.2.1. Let E and E' be (the total spaces of) Banach vector bundles over a base S. Let $f\colon V \to E'$ be a map defined on some open subset $V \subset E$. Then f is said to be *of C^1-class over S* or simply *C^1-differentiable* if the differential on the vertical directions

$$df\colon V \times_S E \to V \times_S E'$$

defines a continuous linear map (in the norm topology: Theorem C.1.1) over the base V.

Lemma C.2.2. *Let $U \times \mathcal{H}$ and $U \times \mathcal{H}'$ be trivial Banach vector bundles over a topological space U. Let f be a map $U \times \mathcal{H} \to U \times \mathcal{H}'$ of C^1-class over U (Theorem C.2.1). Assume that at some point $(x_0, v_0) \in U \times \mathcal{H}$, the vertical derivative*

$$d_{v_0} f_{x_0}\colon \mathcal{H} \to \mathcal{H}'$$

is a (continuous) linear isomorphism. Then f admits a local C^1-inverse near (x_0, v_0), i.e., there exists a local inverse

$$g\colon U_1 \times V' \to U_1 \times V$$

which is of C^1-class over U_1.[15]

Proof. We argue by a completely analogous argument to [Lan95, Theorem 5.2].

We will write the value of the map f at a point $(x, v) \in U \times \mathcal{H}$ as $f_x(v)$. Via the isomorphism $d_{v_0} f_{x_0}$, we may identify $\mathcal{H}'$ with $\mathcal{H}$ and $d_{v_0} f_{x_0}$ with $\mathrm{id}_{\mathcal{H}}$. We may also translate and assume that $v_0 = 0$.

First, consider the map $p_x(v) := (x, v - f_x(v))\colon U \times \mathcal{H} \to U \times \mathcal{H}$, so that $d_0 p_{x_0} = 0$. By the norm-continuity, we find an open neighborhood U_0 of x_0 and a positive $r > 0$ such that $\|d_v p_x\| < \frac{1}{2}$ holds for $x \in U_0$ and $|v| < r$. By the mean value theorem [Lan95, Lemma 4.2], it follows that

$$p_x(v_1) - p_x(v_2) \in \{x\} \times B_{\leq r/2}$$

for $v_1, v_2 \in B_{\leq r}$ and $x \in U_0$. In other words, p maps $U_0 \times B_{\leq r}$ to $U_0 \times B_{\leq r/2}$.

Next, consider the map $q_{x,w}\colon \mathcal{H} \to \mathcal{H}; q_{x,w}(v) := w + v - f_x(v)$. For each $x \in U_0$ and $|w| \leq \frac{r}{2}$, we have

$$|q_{x,w}(v)| \leq |w| + |v - f_x(v)| \leq \frac{r}{2} + \frac{r}{2}$$

for any $v \in B_{\leq r}$. So $q_{x,w}$ is a map from $B_{\leq r}$ to itself. Moreover, $q_{x,w}\colon B_{\leq r} \to B_{\leq r}$ is a contraction since

$$|q_{x,w}(v_1) - q_{x,w}(v_2)| \leq \frac{1}{2}|v_1 - v_2|$$

again by the mean value theorem. Thus, by the fixed point theorem, for every $x \in U_0$ and $|w| \leq B_{\leq r/2}$ there exists a unique $v \in B_{\leq r}$, which will be denoted by $g_x(w)$, such that $w = f_x(v)$.

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[15] We don't use the C^1-structure of the local inverse in this paper.

We then want to conclude that $g\colon U_0 \times B_{\leq r/2} \to B_{\leq r}$ is continuous. We have the following inequalities

$$\begin{aligned}|g_{x_1}(w_1) - g_{x_2}(w_2)| &= |p_{x_1}g_{x_1}(w_1) + f_{x_1}g_{x_1}(w_1) - p_{x_1}g_{x_2}(w_2) - f_{x_1}g_{x_2}(w_2)| \\ &\leq |p_{x_1}(g_{x_1}(w_1)) - p_{x_1}(g_{x_2}(w_2)| + |w_1 - f_{x_1}g_{x_2}(w_2)| \\ &\leq \frac{1}{2}|g_{x_1}(w_1) - g_{x_2}(w_2)| + |w_1 - w_2| + |f_{x_2}(g_{x_2}(w_2)) - f_{x_1}(g_{x_2}(w_2))|\end{aligned}$$

so that

$$|g_{x_1}(w_1) - g_{x_2}(w_2)| \leq 2|w_1 - w_2| + 2|f_{x_2}(v_2) - f_{x_1}(v_2)|$$

and the continuity of g follows from the continuity of $f\colon U \times \mathcal{H} \to \mathcal{H}'$. This completes the proof that f is a local homeomorphism at that point.

We don't prove the continuity of the differential of g because we don't need this. ☐

Corollary C.2.3 (Submersions are smooth)**.** *Let $U \times \mathcal{H}$ and $U \times \mathcal{H}'$ be trivial Banach vector bundles over a topological space U. Let f be a map $U \times \mathcal{H} \to U \times \mathcal{H}'$ of C^1-class over U (Theorem C.2.1). Assume that at some point $(x, v) \in U \times \mathcal{H}$ the vertical differential*

$$d_v f_x \colon \mathcal{H} \to \mathcal{H}'$$

is surjective and that $W_0 \coloneqq \ker(d_v f_x)$ admits a complementary closed linear subspace in $\mathcal{H}$. Then there exist open subsets $U_1 \times V \ni (x, v)$ of $U \times \mathcal{H}$, V' of $\mathcal{H}'$and a homeomorphism$V' \times W_0 \cong V$ such that the diagram

$$\begin{tikzcd} U_1 \times V' \times W_0 \arrow[r, phantom, "\cong"] \arrow[d, "\mathrm{pr}"] & U_1 \times V \arrow[r, phantom, "\subset"] & U \times \mathcal{H} \arrow[dl, "f"] \\ U_1 \times V' \arrow[r, phantom, "\subset"] & U_1 \times \mathcal{H}' & \end{tikzcd}$$

commutes.

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